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THESIS

DIRECT BIT DETECTION RECEIVER NOISE PERFORMANCE ANALYSIS FOR 32-PSK AND 64-PSK MODULATED SIGNALS

by

Iftikhar Ahmed

December 1987

Thesis Advisor: Daniel C. Bukofzer

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Simple two channel receivers for 32-PSK and 64-PSK modulated signals have been proposed which allow digital data (namely bits), to be recovered directly instead of the traditional approach of symbol detection followed by symbol to bit mappings. This allows for binary rather than M-ary receiver decisions, reduces the amount of signal processing operations and permits parallel recovery of the bits. The noise performance of these receivers quantified by the Bit Error Rate (BER) assuming an Additive White Gaussion Noise interference model is evaluated as a function of E_b/N_o, the signal to noise ratio, and transmitted phase angles of the signals. The performance results of the direct bit detection receivers (DBDR) when compared to that of conventional phase measurement receivers demonstrate

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Direct Bit Detection Receiver Noise Performance Analysis for 32-PSK and 64-PSK Modulated Signals

by

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ABSTRACT

Simple two channel receivers for 32-PSK and 64-PSK modulated signals have been proposed which allow digital data (namely bits), to be recovered directly instead of the traditional approach of symbol detection followed by symbol to bit mappings. This allows for binary rather than M-ary receiver decisions, reduces the amount of signal processing operations and permits parallel recovery of the bits. The noise performance of these receivers quantified by the Bit Error Rate (BER) assuming an Additive White Gaussion Noise interference model is evaluated as a function of E_b/N_o, the signal to noise ratio, and transmitted phase angles of the signals. The performance results of the direct bit detection receivers (DBDR) when compared to that of conventional phase measurement receivers demonstrate that DBDR's are optimum in BER sense. The simplicity of the receiver implementations and the BER of the delivered data make DBDR's attractive for high speed, spectrally efficient digital communication systems.

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LIST OF ABBREVIATIONS

A	Amplitude of Signal
AWGN	Additive White Gaussian Noise
BER	Bit Error Rate
BPSK	Binary Phase Shift Keying
BW	Bandwidth
dBs	Decibels
DBD.	Direct Bit Detection
DBDR	Direct Bit Detection Receiver
E_b	Bit Enregy
E_s	Symbol Energy
f_c	Carrier Frequency
MASK	M-ary Amplitude Shift Keying
MFSK	M-ary Frequency Shift Keying
MPSK	M-ary Phase Shift Keying
MSB	Most Significant Bit
N_o	Noise Power Spectral Density Level
n_i	Component of Noise Along Dimension $\varphi_i(t)$, $i = 1,2$
p.d.f	Probability Density Function
PE	Probability of Error
PSK	Phase Shift Keying
Q(.)	Q-Function of Argument (.)
QPSK	Quadrature Phase Shift Keying
r_i	Component of the Received Signal Along Dimension $\varphi_i(t)$, $i = 1,2$
RF	Radio Frequency
$R(\tau)$	Correlation Function
SER	Symbol Error Rate
S(f)	Power Spectral Density
$s_i(t)$	Transmitted Signals, $i = 1,2,----64$
SNR	Signal-to-Noise Ratio
T_b	Bit Duration in Seconds
T_s	Symbol Duration in Seconds

α, β, γ and δ	Signal Phase Angles
η	Phase Angle of the Received Signal
θ_m	True Phase of the Signal
θ_o	Initial Phase of the Signal
$\varphi_i(t)$	Basis Funtions, $i = 1,2$

I. INTRODUCTION

Digital carrier modulation is required to transmit baseband digital information over a bandpass channel. The digital information is assumed to be binary, that is, '0' and '1' and occur at a rate of $1/T_b$ bps. Two unique signals are needed to represent these digits.

The binary digits can be segmented into blocks where each block consists of k-bits. Since there are $M = 2^k$ such distinct blocks, M different signals are required to represent the k-bit blocks unambiguously. Each k-bit block is called a symbol and the symbol duration is $T_s = k\ T_b$ seconds. This type of digital transmission is called M-ary signalling.

A wide variety of M-ary signalling techniques have been discussed in the literature. These techniques include M-ary Amplitude Shift Keying (MASK), M-ary Frequency Shift Keying (MFSK), M-ary Phase Shift Keying (MPSK) and composite modulation techniques involving combinations of amplitude, frequency and phase shift keying. Of these techniques, however, only a limited subset is desirable for specific digital communication applications. The transponder nonlinearities and power-efficiency requirements in satellite communication, for instance, usually constrain the modulation format to have a constant envelope. A commonly used digital modulation technique in satellite communication applications is MPSK, where the modulated signal has a constant envelope and utilizes relatively small amounts of bandwidth. In this case, M different phases of the carrier are used to represent the M distinct symbols that make up a k-bit block of binary digits.

At present, BPSK and QPSK are among the most common modulation techniques in practical applications. Higher order MPSK modulation techniques are spectrally more efficient, however they incur severe signal-to-noise ratio penalties. With the growing constraint on the available spectral bandwidth and with new developments in more efficient high power microwave amplifier designs as well as advancements in solid state technology, it may be possible to design communication systems which operate at large enough signal-to-noise ratios, so as to enable the use of the more spectrally efficient modulation techniques.

In MPSK modulated signals, each symbol corresponds to a unique phase of the carrier, and the receiver extracts symbols based on the estimation of the received signal's phase angle. In the receiver, some method must be implemented to regenerate bits from the estimated symbol phases.

A novel design using only two channel outputs which are produced by correlation of the received signal with an unmodulated sine and cosine reference signal has been investigated for 8-PSK modulated signals in Reference 1, and extended to 16-PSK modulated signals in Reference 2. These designs have the advantage of recovering bits directly by a technique refered to as the Direct Bit Detection (DBD) method. This method employs simple receiver logic to recover each bit of a symbol separately, by further processing of the correlator outputs. The direct recovery of the bits not only dispenses with the circuitry necessary for the symbol-to-bit mapping in the receiver, but also provides a simple mathematical approach enabling the calculation of the receiver's bit error rate directly, thus providing a better measure of system performance from the user's point of view.

This thesis further extends the DBD method to receivers processing 32-PSK and 64-PSK modulated signals. In Chapter II, some basic background information is provided in connection with M-ary PSK signalling, with emphasis on the concept of signal representations using signal space vector diagrams and on conventional receiver designs.

In Chapter III, a two channel receiver for 32-PSK modulated signals is presented. The logic needed to recover each bit separately is explained with the help of the signal space diagram. The noise performance of the receiver is analyzed for each bit of the 5-bit block, and closed form expressions are obtained for the overall bit error rate in terms of signal-to-noise ratio and the transmitted signal phase angles. These bit error rate expressions were evaluated on the computer and plotted as a function of signal to noise ratio. The composite receiver bit error rate values were then compared with those of conventional 32-PSK receivers.

Chapter IV addresses for 64-PSK modulated signals the same issues analyzed in Chapter III for DBDR's processing 32-PSK modulated signals. Results here are presented in the form similar to those presented in Chapter III.

In Chapter V detailed comparisons of the performance of these receivers with that of standard phase measurement receivers are carried out, and issues dealing with implementational advantages as well as disadvantages are discussed.

II. DEMODULATION OF M-ARY PHASE SHIFT KEYED SIGNALS

A. SIGNAL REPRESENTATION OF M-ARY PHASE SHIFT KEYED MODULATION

The transmitter for MPSK modulated signals maps blocks of input bits into M distinct phases of a carrier. For k input bits per block, there are $2^k = M$ combinations for which a one-to-one correspondence with the carrier phase position is made. For example, if three bits at the modulator input are taken at a time, then $2^3 = 8$ phase positions result producing 8-PSK modulated signals. The simplest forms of MPSK modulation are Binary Phase Shift Keyed (BPSK) and Quadrature Phase Shift Keyed (QPSK) modulations.

In BPSK, the phase of a carrier is switched between two values according to the two possible data bits '1' and '0'. The two phases are usually separated by π radians, so that the two possible trasmitted signals can be written as

$$s_1(t) = A\cos(2\pi f_c t + \theta_o) \qquad 0 \le t \le T_b \qquad \text{when '1' is sent} \tag{2.1}$$

$$\begin{aligned} s_2(t) &= A\cos(2\pi f_c t + \theta_o + \pi) \qquad 0 \le t \le T_b \qquad \text{when '0' is sent} \\ &= -A\cos(2\pi f_c t + \theta_o) \end{aligned} \tag{2.2}$$

where

A = Signal amplitude

f_c = Carrier frequency

θ_o = Initial phase of the carrier

T_b = Bit duration in seconds

These two carrier states can be represented by Figure 2.1(a) .

In QPSK, $M=4$, so that $2^k = 4$ implies that $k=2$, and therefore 2 bits form a symbol with the symbol duration being twice that of the bit duration, that is, $T_s = 2T_b$. The carrier phase now shifts in increments of $\pi/2$ radians [3], so that letting H_1, H_2, H_3 and H_4 be the four possible symbols consisting of two bits, the corresponding waveforms can be written as

$$s_1(t) = A\cos(2\pi f_c t + \theta_o + \pi/4) \qquad 0 \le t \le T_s \qquad \text{when } H_1 \text{ is sent} \tag{2.3}$$

$$s_2(t) = A\mathrm{sos}(2\pi f_c t + \theta_o + 3\pi/4) \qquad 0 \le t \le T_s \qquad \text{when } H_2 \text{ is sent} \tag{2.4}$$

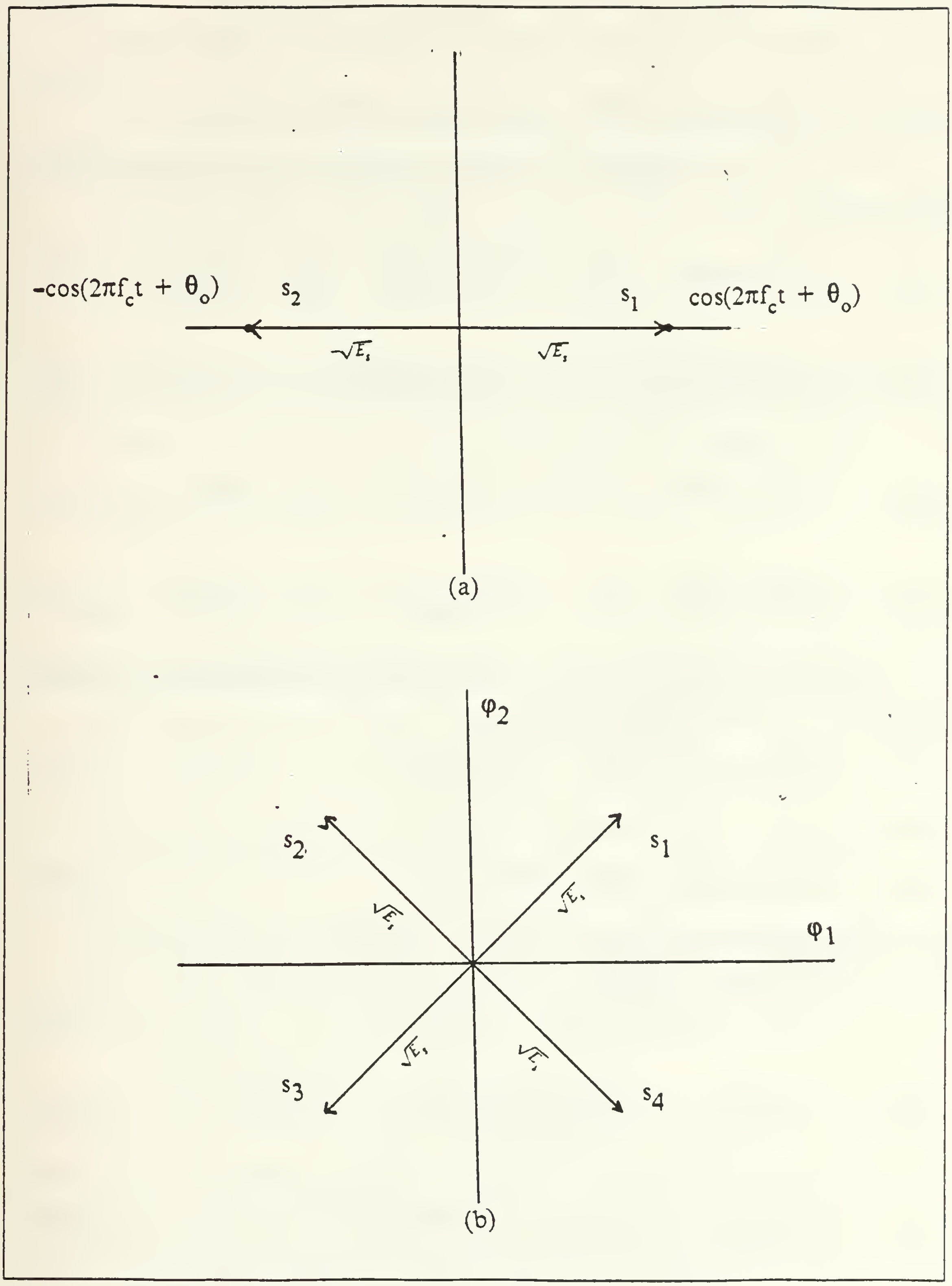

Figure 2.1 Signals for (a) BPSK and (b) QPSK Modulations.

$$s_3(t) = A\cos(2\pi f_c t + \theta_o + 5\pi/4) \quad 0 \leq t \leq T_s \quad \text{when } H_3 \text{ is sent} \tag{2.5}$$

$$s_4(t) = A\cos(2\pi f_c t + \theta_o + 7\pi/4) \quad 0 \leq t \leq T_s \quad \text{when } H_4 \text{ is sent} \tag{2.6}$$

If we expand Equations 2.3 - 2.6 using trignometeric identities, we can further write them as

$$s_1(t) = (A/\sqrt{2})\cos(2\pi f_c t + \theta_o) + (A/\sqrt{2})\sin(2\pi f_c t + \theta_o) \quad 0 \leq t \leq T_s \tag{2.7}$$

$$s_2(t) = -(A/\sqrt{2})\cos(2\pi f_c t + \theta_o) + (A/\sqrt{2})\sin(2\pi f_c t + \theta_o) \quad 0 \leq t \leq T_s \tag{2.8}$$

$$s_3(t) = -(A/\sqrt{2})\cos(2\pi f_c t + \theta_o) - (A/\sqrt{2})\sin(2\pi f_c t + \theta_o) \quad 0 \leq t \leq T_s \tag{2.9}$$

$$s_4(t) = (A/\sqrt{2})\cos(2\pi f_c t + \theta_o) - (A/\sqrt{2})\sin(2\pi f_c t + \theta_o) \quad 0 \leq t \leq T_s \tag{2.10}$$

Vector space representation of the signals [Ref. 4 pp. 188-190] can be utilized by observing that two basis functions given by

$$\varphi_1(t) = \sqrt{(2/T_s)}\cos(2\pi f_c t + \theta_o) \quad 0 \leq t \leq T_s \tag{2.11}$$

$$\varphi_2(t) = \sqrt{(2/T_s)}\sin(2\pi f_c t + \theta_o) \quad 0 \leq t \leq T_s \tag{2.12}$$

allow the expression of signals $s_i(t)$, $i = 1,2,3,4$ in the form

$$s_1(t) = \sqrt{(E/2)}\varphi_1(t) + \sqrt{(E/2)}\,\varphi_2(t) \quad 0 \leq t \leq T_s \tag{2.13}$$

$$s_2(t) = -\sqrt{(E/2)}\varphi_1(t) + \sqrt{(E/2)}\,\varphi_2(t) \quad 0 \leq t \leq T_s \tag{2.14}$$

$$s_3(t) = -\sqrt{(E/2)}\varphi_1(t) - \sqrt{(E/2)}\,\varphi_2(t) \quad 0 \leq t \leq T_s \tag{2.15}$$

$$s_4(t) = \sqrt{(E/2)}\varphi_1(t) - \sqrt{(E/2)}\,\varphi_2(t) \quad 0 \leq t \leq T_s \tag{2.16}$$

where it can be easily proved that $\varphi_1(t)$ and $\varphi_2(t)$ form a complete orthonormal set of functions.

Observe that E is the energy of each signal in which $E = A^2T_s$ for $s_i(t)$, $i = 1,2,3,4$.

All signals can be represented by vectors in a two dimensional signal space, in which all together the vectors form a "Signal Constellation".

For M-ary Phase Shift Keyed modulation (with equal signal energies) a convenient representation of the signal set is given by

$$s_i(t) = \sqrt{(2E_s/T_s)}\cos(2\pi f_c t + 2\pi(i-1)/M) \quad 0 \le t \le T_s \quad i = 1,2,3,\text{----}\ M \tag{2.17}$$

where it is assumed that $2\pi f_c$ is an integer multiple of $2\pi/T_s$ and the initial phase θ_0 can be set to zero without loss of generality. As previously stated, a suitable complete orthonormal set of functions for the representation of MPSK signals is given by

$$\varphi_1(t) = \sqrt{(2/T_s)}\cos(2\pi f_c t) \quad 0 \le t \le T_s \tag{2.18}$$

$$\varphi_2(t) = \sqrt{(2/T_s)}\sin(2\pi f_c t) \quad 0 \le t \le T_s \tag{2.19}$$

so that Equation 2.17 can be written as

$$s_i(t) = \sqrt{E_s}\cos(2\pi(i-1)/M)\ \varphi_1(t) - \sqrt{E_s}\sin(2\pi(i-1)/M)\ \varphi_2(t) \tag{2.20}$$
$$0 \le t \le T_s \qquad i = 1,2,3,\text{-----},M$$

The signal constellations for QPSK are shown in Figure 2.1(b), for 8-PSK in figure 2.2(a) and for 16-PSK in Figure 2.2 (b) respectively.

B. POWER SPECTRAL DENSITY OF MPSK MODULATED SIGNALS

The power spectral density of MPSK modulated signals with an M-level modulating baseband waveform of duration T_s is given by [5]

$$S(f) = \frac{E_s}{2}\left[\left[\frac{\sin\pi(f-f_c)T_s}{\pi(f-f_c)T_s}\right]^2 + \left[\frac{\sin\pi(f+f_c)T_s}{\pi(f+f_c)T_s}\right]^2\right] \tag{2.21}$$

where E_s is the symbol energy given by $A^2T_s/2$, $T_s = kT_b$, where k is equal to $\log_2 M$. Equation 2.21 is plotted in Figure 2.3 for $f > 0$.

From Figure 2.3 it is clear that the width of the main spectral lobe is $2/T_s$ Hz. so that the minimum signal bandwidth is estimated by $2/T_s$ Hz.

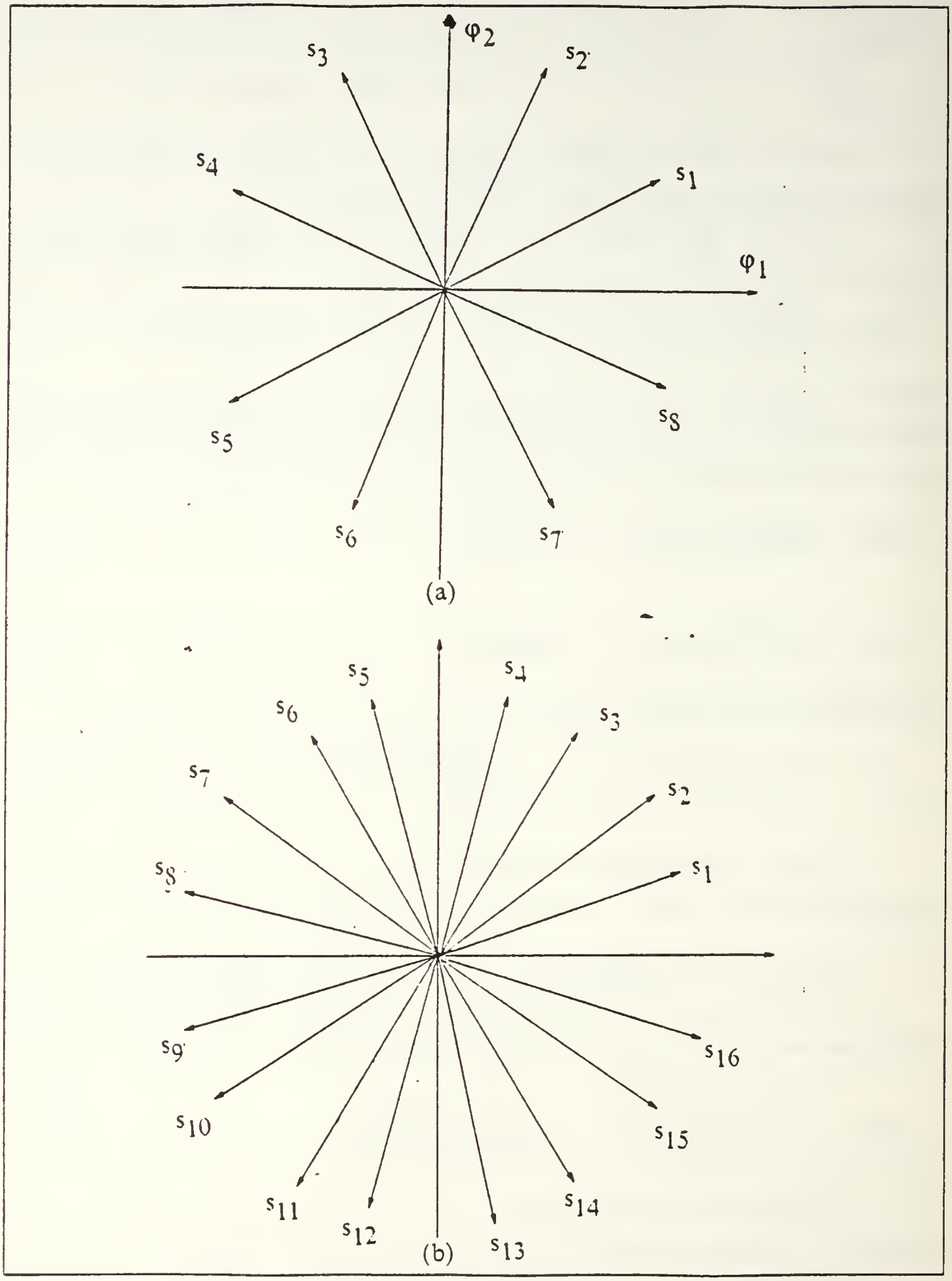

Figure 2.2 Signal Constelation for (a) 8-PSK and (b) 16-PSK Modulated Signals.

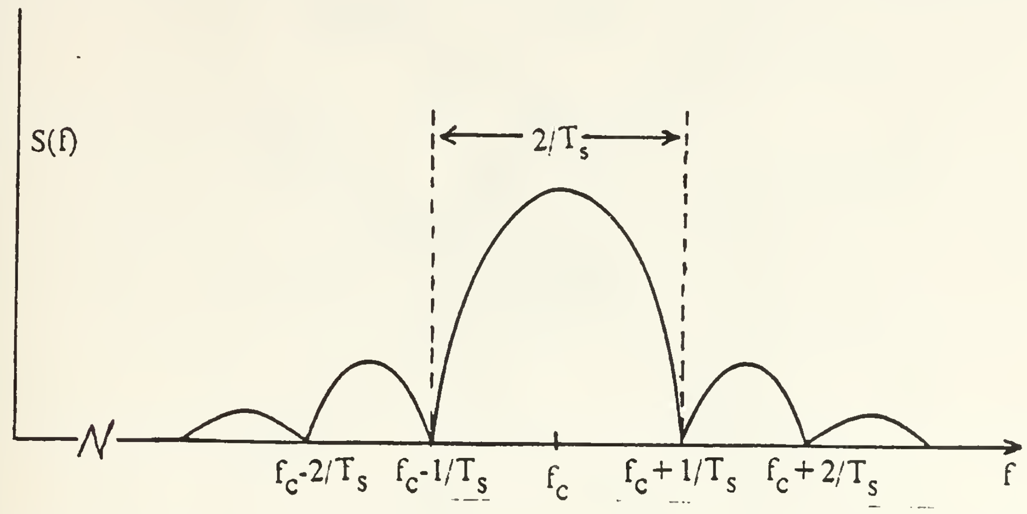

Figure 2.3 Normalized Power Spectrum of a MPSK Modulated Signal.

As bandwidth limitations become more severe and more signal power is available due to improved amplifiers, MPSK offers some real performance advantages due to its bandwidth efficiency. If bandwidth is a more severe constraint than signal power, larger values of M, that is, M equal to 8, 16, 32 or 64 become attractive since the symbol rate, hence the bandwidth, decreases as $1/\log_2 M = 1/k$. Furthermore the signal bandwidth requirements for a given value of M can be reduced by appropriate signal filtering and equalization [3].

C. ENCODING OF SYMBOLS

At the receiver, errors in demodulation are made when the detected phase of the received waveform, which consists of signal and noise, differ from predetermined limits about the actual transmitted signal phase. The limits are generally between $+\pi/M$ and $-\pi/M$ about the actual transmitted signal phase as shown in Figure 2.4.

In general, the most likely receiver errors involve mistaking a correct symbol with the immediately adjacent symbol. In order to reduce the probability of receiver error, adjacent symbols are chosen so they differ by only one bit, which results in the so-called Gray-Code mapping. In the case of the Direct Bit Detection method, which is the focus of this thesis and is dealt with in detail in the sequel, it will be seen that Gray-Code mapping is still useful as it results in the simplest logic which allows the

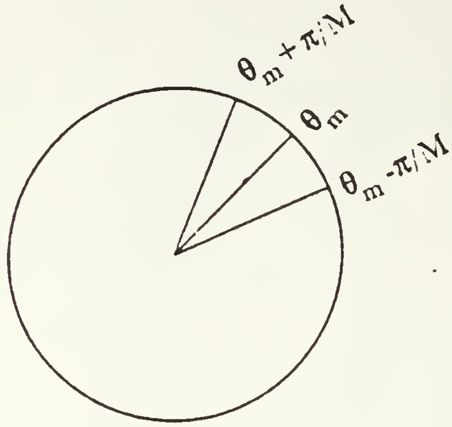

Figure 2.4 Decision Regions for Coherent Detection of MPSK Modulated Signals.

detection of a particular bit. The signal constellations of 8-PSK and 16-PSK with Gray-Code mappings are shown in Figure 2.5(a) and 2.5(b) respectively.

D. OPTIMUM COHERENT DEMODULATION

In order to detect one out of M possible transmitted signals in an M-ary signalling scheme with minimum probability of error, the optimum demodulator must be synchronized with the transmitter's unmodulated carrier. In other words the initial phase of the transmitted signal must be perfectly known at the receiver. Detailed mathematical treatments of this subject matter is available in many texts [ref. 5 pp. 386-388]. The basic form of the receiver is the one in which the received signal r(t) is correlated with each of the elements of the orthonormal signal set followed by appropriate weighting and summing of the correlator outputs. The overall receiver structure is known as a "Correlation Receiver". Coherent demodulation of MPSK modulated signals implies in principle the use of M signal processing channels in the receiver. This kind of receiver is shown in Figure 2.6(a). A second method of implementing the optimum demodulator is to apply the received signal r(t) through M parallel filters with impulse responses

$$h_i(t) = s_i(T_s - t) \qquad i = 1,2,3,----,M \qquad 0 \leq t \leq T_s \tag{2.22}$$

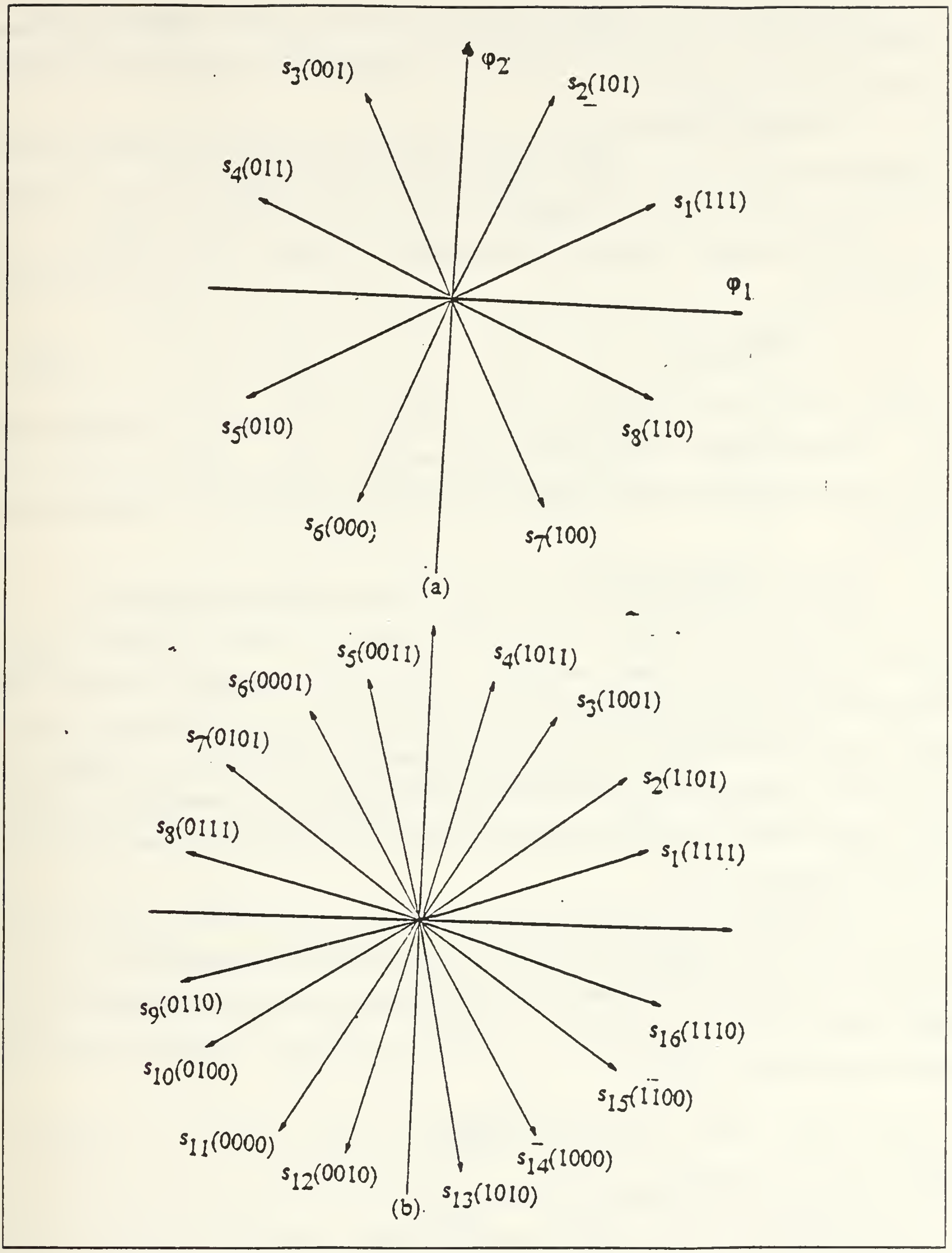

Figure 2.5 Signal Constellation for (a) 8-PSK and (b) 16-PSK Modulated Signals with Gray Code Bit Assignments.

with the remainder of the receiver being the same as the above described correlation receiver [4]. Such a deomdulator is known as a Matched Filter receiver whose structure is shown in Figure 2.6(b).

For MPSK modulated signals, the received signal can be demodulated using two channels known as the Inphase (I) and Quadrature (Q) channels. A complete mathematical treatment of such a receiver is given in reference 5 pp. 425-428.

The demodulator makes a decision based on an estimate of the transmitted phase computed as

$$\theta_m = \mathrm{Arctan}(r_2/r_1) \tag{2.23}$$

where r_1 and r_2 are the in-phase and the quadrature components of the received signal r(t). A correct decision is made if the estimated value of θ_m is within $+\pi/M$ or $-\pi/M$ of the transmitted value of θ_m. This kind of receiver is shown in block diagram form in Figure 2.7.

E. SYMBOL ERROR PROBABILITY AND BIT ERROR PROBABILITY

All receivers described in the previous section for MPSK modulated signals recover the symbols by phase estimation. However, for meaningful recovery of the original data, symbol to bit mappings have to be carried out at the receiver in order to deliver binary information to the intended user(s). Symbol error probabilities or symbol error rates (SER) can be computed directly from the knowledge of the signal and channel characteristics and such derivations yielding closed form expressions can be found in many texts [5].

The interference model used in these analyses is the so-called Additive White Gaussian Noise (AWGN) model. The power spectral density of this Gaussian noise is essentially constant up to frequencies much higher than those that are significant in the signal. Such a noise is implied to be a stationary, zero mean Gaussian process with power spectral density given by

$$S_n(f) = N_o/2 \quad \text{and} \quad R_n(\tau) = (N_o/2)\,\delta(\tau) \tag{2.24}$$

where $R(\tau)$ is the corresponding correlation function.

This interference model will be used in the noise performance analyses of Direct Bit Detection receivers for 32-PSK and 64-PSK modulated signals in the next two chapters. The results will be contrasted with the performance of corresponding conventional receivers of the type presented in the previous section. The performance

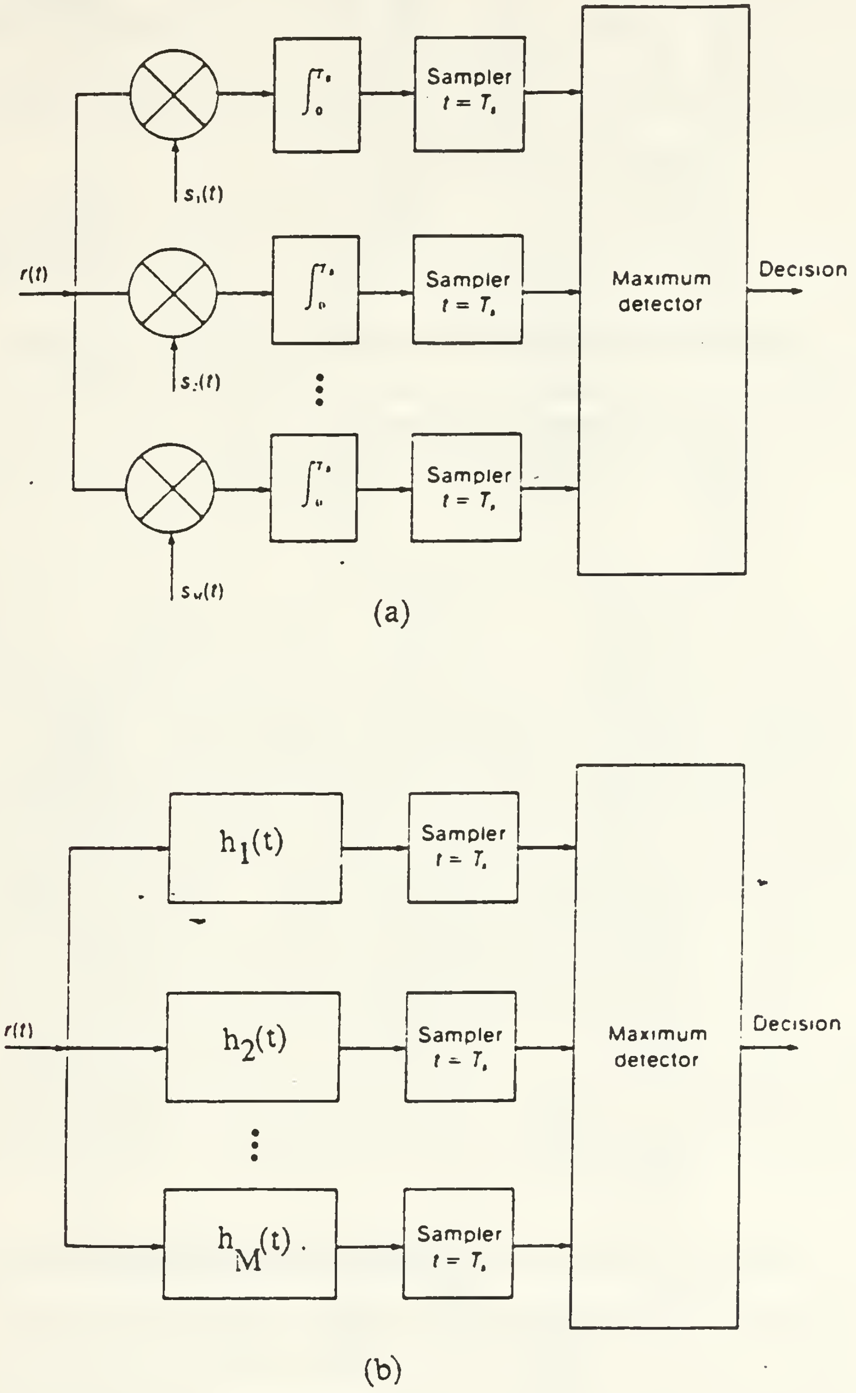

Figure 2.6 (a) Correlation Demodulator (b) Matched Filter Demodulator.

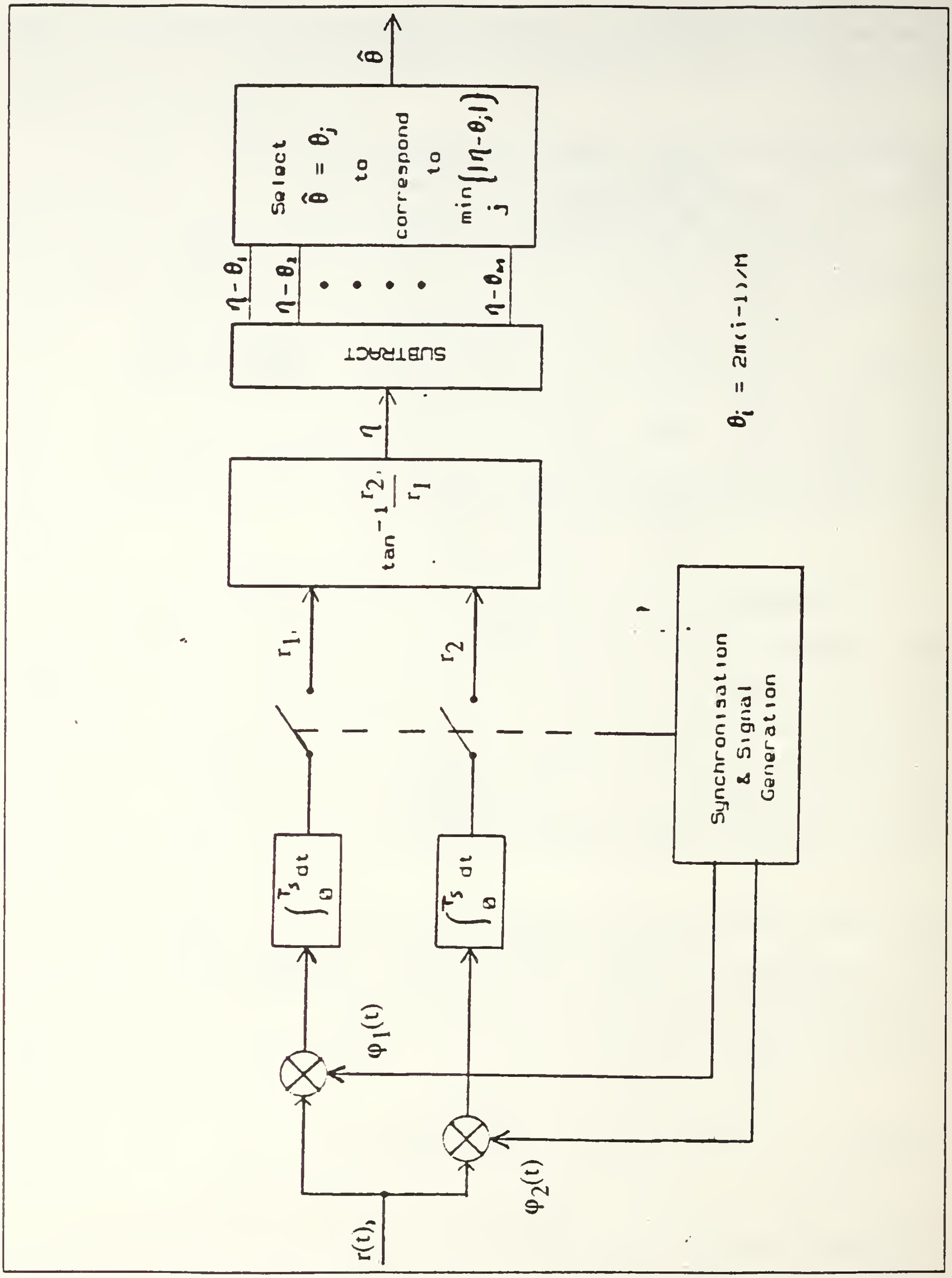

Figure 2.7 A Two Channel Demodulator for MPSK Modulated Signals.

in SER sense of such receivers has been derived in Reference 5 [pp. 425-428] and the probability P_s of symbol error is approximately given by

$$P_s \sim 2Q(\sqrt{(2E_s/N_0)} \sin \pi/M) \quad \text{for } M \geq 4 \tag{2.25}$$

where E_s is symbol energy, and $E_s = kE_b$ where E_b is the energy per bit and while Q(.) is defined by

$$Q(w) = \frac{1}{\sqrt{2\pi}} \int_w^\infty \exp(-u^2/2)\, du \tag{2.26}$$

When a Gray-Code mapping is used, two symbols that correspond to adjacent carrier phases differ by only 1 bit. Hence when the demodulator incorrectly selects the adjacent phase for the true phase, only one bit of k-bits is in error. Therefore, the average probability of bit error denoted by P_b can be approximated by

$$P_b \sim P_s/k \tag{2.27}$$

It must be pointed out that symbols are created for the benefit of communications engineers, and are transparent to the channel user. The focus from a customer point of view is the Bit Error Rate (BER). Also, when comparing systems with different levels of modulation, the bit error rate rather than symbol error rate is of greater interest. The computation of bit error rate is often quite complicated for multilevel modulation systems . The bit error rate depends on both the symbol error rate, the type of symbol error and the way in which a block of bits are grouped together to form the symbol waveforms prior to modulation of the carrier.

Closed form expressions of the Bit Error Rate for MPSK when a Gray-Code mapping is used for encoding the symbols have been obtained in Reference 6. Similiar results for QPSK, 8-PSK and 16-PSK have also been derived independently in Reference 7.

F. DIRECT BIT DETECTION OF MPSK MODULATED SIGNALS

As has been previously stated, the recovery of digital data at the receiver requires a method for separating bits from symbols if more than one bit is mapped to each symbol. While such inverse mapping from symbols to bits can be implemented by various methods, in direct bit detection (DBD) receivers each bit of the transmitted symbol is recovered separately. This is done by implementing additional logic after the sine and cosine components of the received signal are obtained via correlation operations.

A direct bit detection receiver is in a sense an extension of the bit recovery scheme used for QPSK [Ref. 4, pp. 120-124], in which the bits are recovered directly. The receiver structure for QPSK with Gray-Code bit-to-symbol mapping and corresponding signal constellation diagram is shown in Figure 2.8. Note that this receiver structure is same as the corellator receiver structure introduced in Section D of this chapter.

This methodology has been further extended to the design of direct bit detection receivers for 8-PSK and 16-PSK, as explained in detail in References 7, 1 and 2. The signal constellation diagram for 8-PSK with Gray-Code bit assignments is shown in Figure 2.5(a) and the corresponding DBD receiver is shown in Figure 2.9. Its performance given by PE, the probability of a bit error, as a function of signal-to-noise ratio has been plotted in Figure 2.10. The signal constellation diagram for 16-PSK with Gray-Code bit assignments is shown in Figure 2.5(b) and the DBD receiver block diagram structure is shown in Figure 2.11. Detailed mathematical treatment of the noise performance of this receiver is found in [2] , and its probability of error as a function of signal to noise ratio is plotted in Figure 2.12 .

The above results have become the basis for extension of these techniques to direct bit detection methods for 32-PSK and 64-PSK modulated signals. Detailed derivations on receiver performance and evaluations of bit error rate as a function of signal-to-noise ratio are the main topic of this thesis with the bulk of the work presented in the next two chapters. (Principally based on the work of Reference 2)

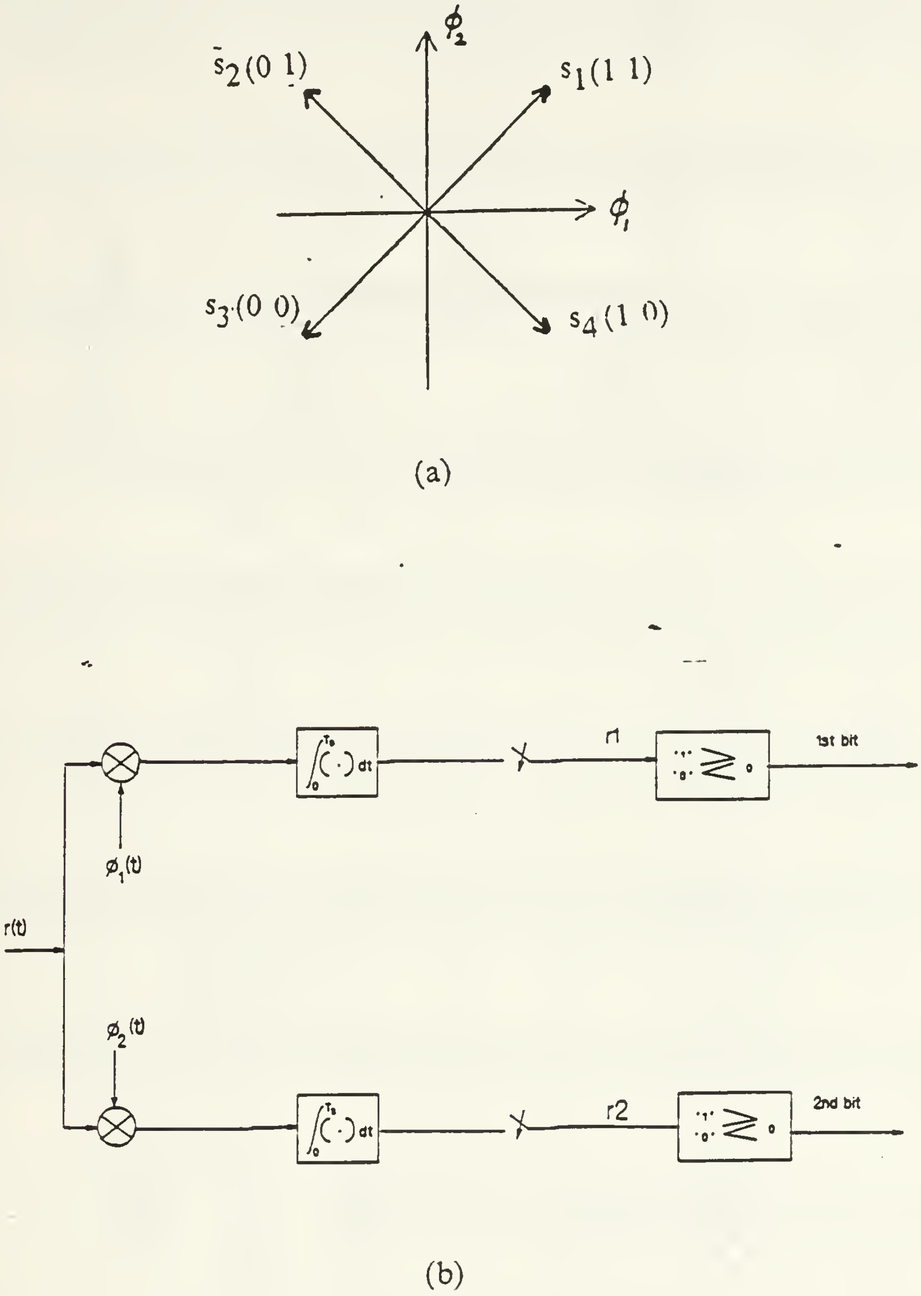

Figure 2.8 QPSK Signal Constellation with Corresponding DBD Receiver.

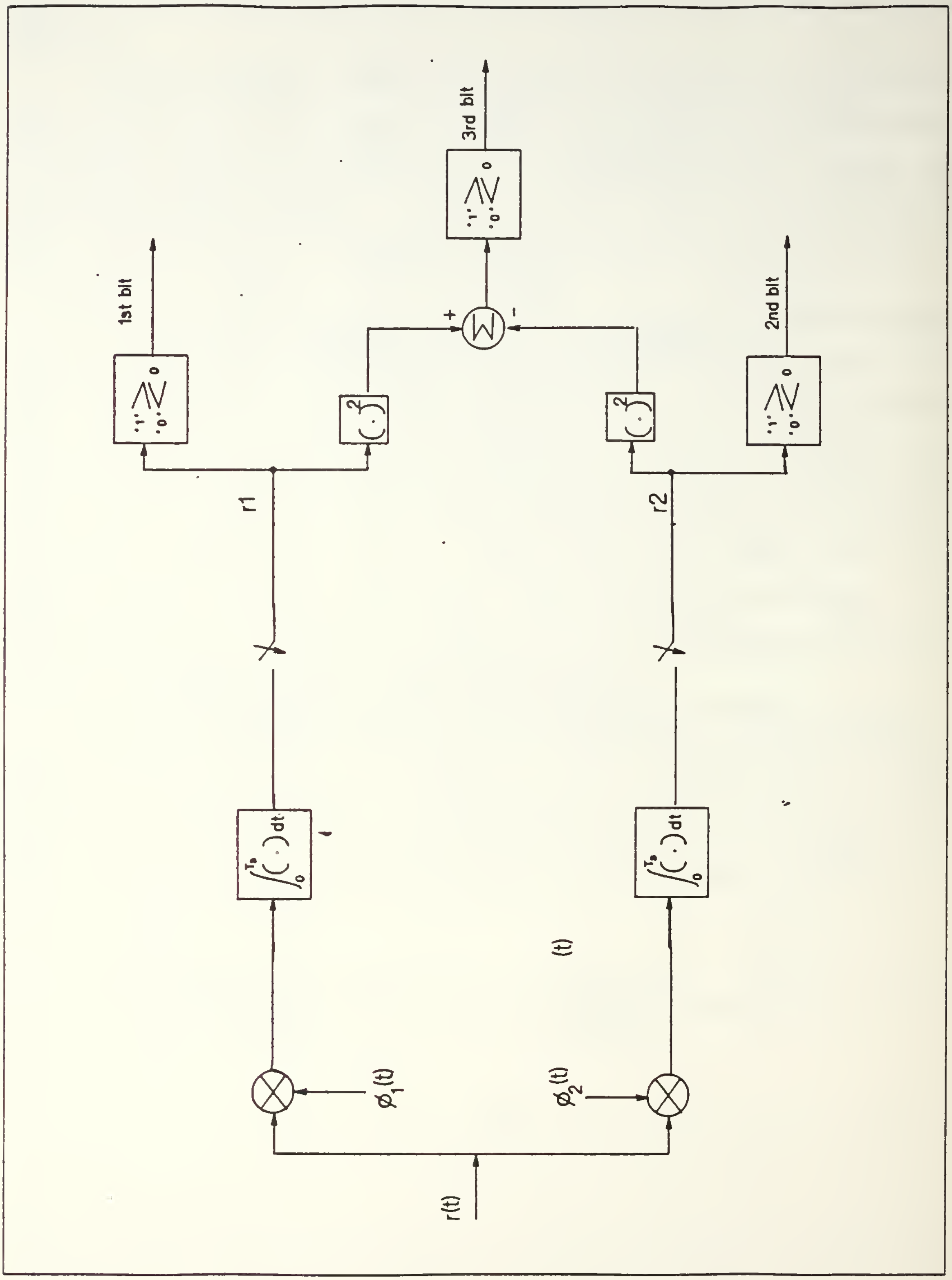

Figure 2.9 DBD Receiver Structure for 8PSK Modulated Signals.

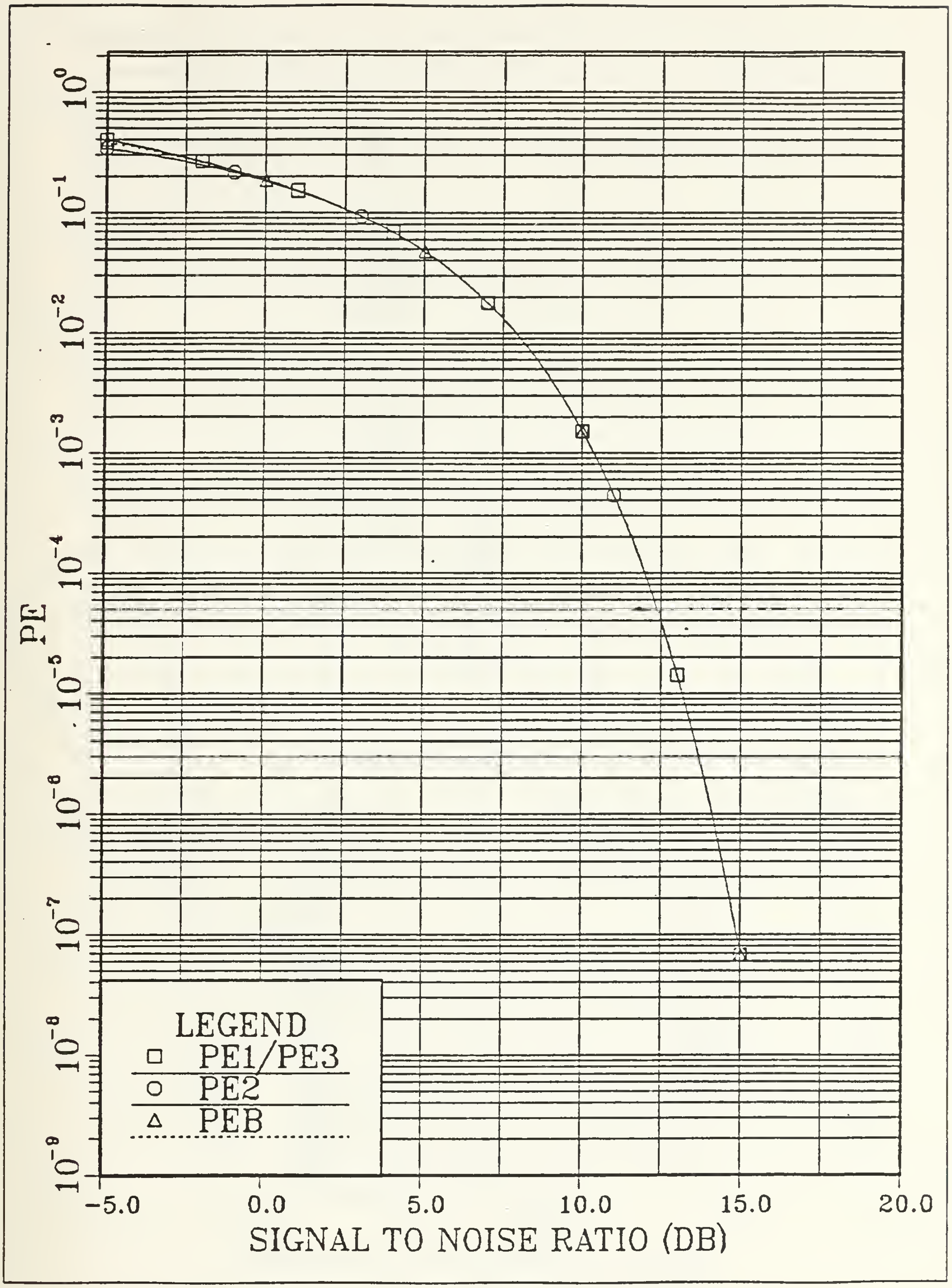

Figure 2.10 Noise Performance Curves for Receiver for 8-PSK Modulated Signals.

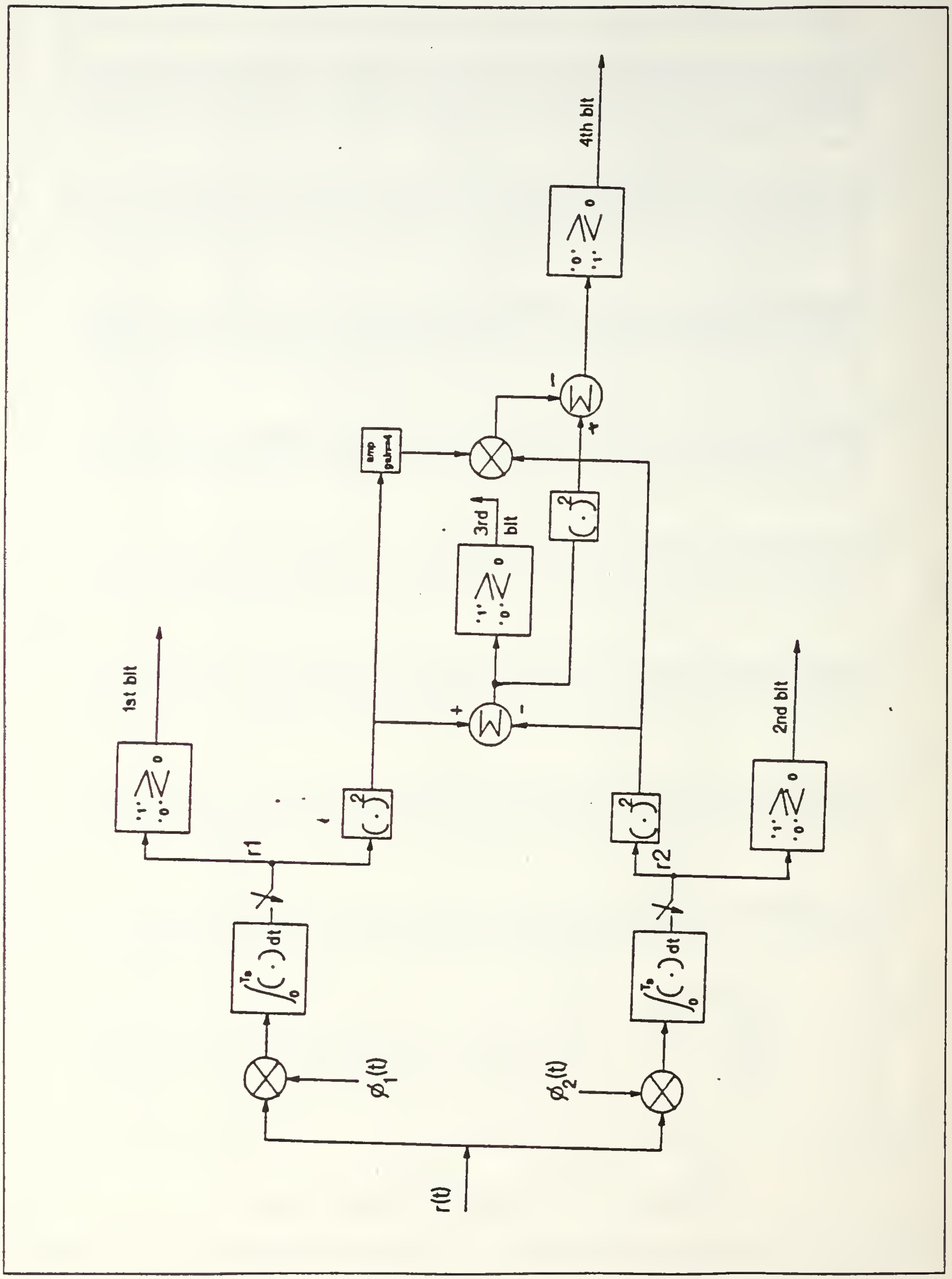

Figure 2.11 Proposed DBD Receiver Structure for 16-PSK Modulated Signals.

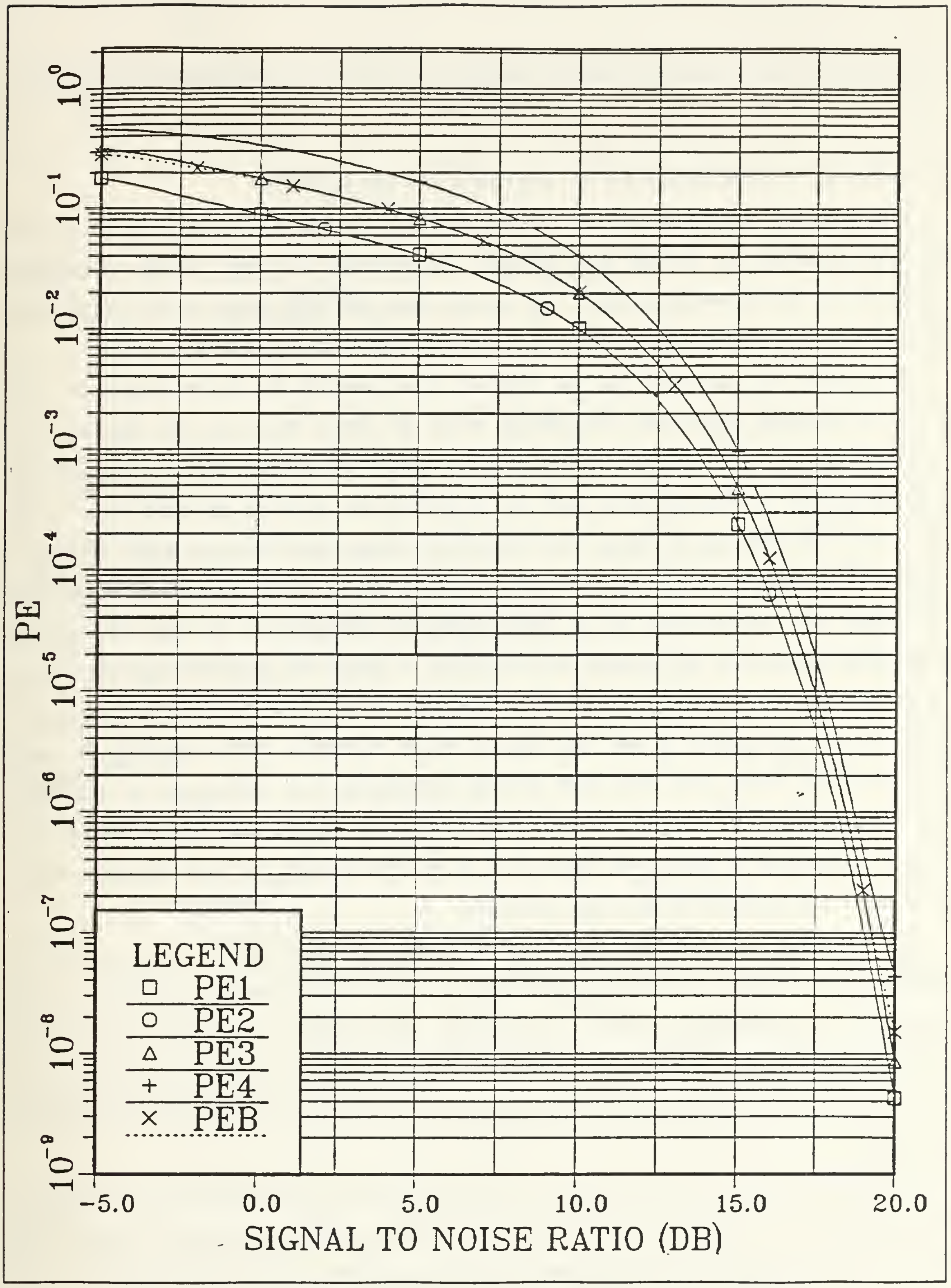

Figure 2.12 Noise Performance Curves for Receiver for 16-PSK Modulated Signals.

III. DIRECT BIT DETECTION RECEIVER FOR 32-PSK MODULATED SIGNALS

A. SIGNAL CONSTELLATION AND RECEIVER DESIGN

The receiver structures and the results obtained for the performance of Direct Bit Detection receivers for 8-PSK and 16-PSK suggest that for any MPSK modulated signal, a direct bit detection receiver will always exist and that under certain conditions, their performance will be same as that of standard phase measurement receivers.

By careful bit mappings to the symbols, it is possible to derive simple receiver logic involving the correlator outputs in order to make decisions for the separate recovery of each bit. It has been explained in the previous chapter that the best scheme for mapping bits to the symbols such as to reduce the average bit error rate is the Gray-Code. The concept of signal representations using basis funtions $\varphi_1(t)$ and $\varphi_2(t)$ forming the signal space has also been introduced in the previous chapter. It is possible to combine the advantages of a Gray-Code bit assignment to the 32 signals, representing 5 bits each, and simple decision rules to derive the receiver logic which can be implemented by simple hardware. First take the Gray-Code mapping of 5 bits (32 combinations) and place it on the signal space diagram while rotating it until assignments are found that will meet certain conditions that will result in the DBD method. An example of a signal constellation for 32-PSK modulated signals with Gray-Code bit assignment is shown in Figure 3.1. A receiver design can now be undertaken which will enable parallel or direct bit recovery.

As has been seen in the case of 8-PSK and 16-PSK [Refs. 1,2], zero value thresholds and simple arithmetic operations like multiplication, addition, and subtraction are required to make bit decisions. By careful examination of the bit assignment for 32-PSK modulated signals (Figure 3.1), it can be seen that for such signals the first four bits can be recovered in the same way as in the 16-PSK modulated signals case.

The decision rule for the recovery of all the bits can be derived from the signal space diagram. No interference will be considered while defining the receiver logic. The effect of noise on receiver performance will be analyzed in the next section. The decision for the most significant bit (MSB), which will be synonymously referred to as the first bit, is based on the component of the signals along the φ_1 axis. All the

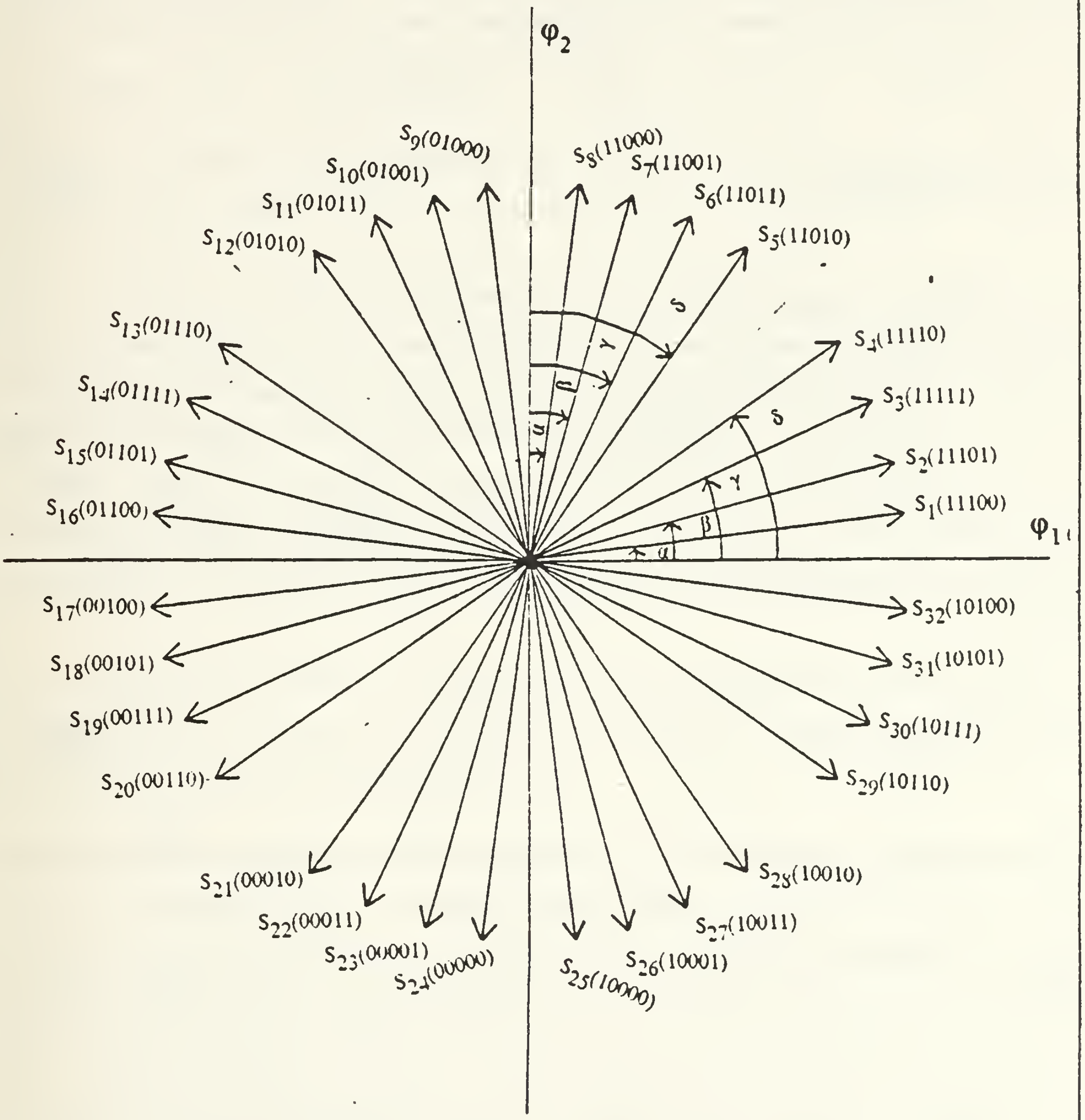

Figure 3.1 Signal Constellation for 32-PSK Modulated Signals with Gray Code Bit Assignments.

symbols for which first bit is a '1' have projections on the positive X-axis and all the symbols with a first bit of '0' have negative projections on the X-axis. Thus, correlating the received signal coherently with the function $\varphi_1(t)$ results in a constant positive voltage whenever the transmitted symbol has a 1st bit of a '1' and a negative voltage if the 1st bit is a '0' in the absence of any interference. That means decisions can be made based on whether $r_1 > 0$ or $r_1 < 0$, where r_1 is defined as

$$r_1 = \int_0^{T_s} r(t)\, \varphi_1(t)\, dt \tag{3.1}$$

Considering the second bit, it can be observed from Figure 3.1 that all the symbols for which the second bit is a '1' have projections on the positive Y-axis (or the φ_2 axis) and the symbols for which the 2nd bit is a '0' have projections on the negative Y-axis. Hence, coherently correlating the received signal with the function $\varphi_2(t)$ will result in a constant positive value for the symbols with a 2nd bit of '1' and a negative value for symbols with a 2nd bit of a '0', assuming no interference. Thus a decision on the 2nd bit can be implemented by testing whether $r_2 > 0$ or $r_2 < 0$, where r_2 is defined as

$$r_2 = \int_0^{T_s} r(t)\, \varphi_2(t)\, dt \tag{3.2}$$

Focusing now on the third bit, it is observed that the symbols for which the third bit is a '1', in an absolute value sense, have projections on the X-axis that are large while the corresponding projections on the Y-axis are small. The converse is true for the symbols for which the third bit is a '0'. Squaring these projection values rather than taking their absolute values has the advantage of further separating large values from small values. Thus by squaring the outputs of the two correlators (which yield these projections) and taking their difference provides a bipolar output (in the absence of interference) which allows for a decision on the third bit. In notational terms this implies that decisions on the 3rd bit can be made based on whether $(r_1^2 - r_2^2) > 0$ or $(r_1^2 - r_2^2) < 0$.

For the fourth bit, decisions are made on the basis of the value of the received phase angle η, where $\eta = \tan^{-1}(r_2/r_1)$ even though, as will be demonstrated, no phase angle measurements are actually made. For any reasonable choice of signal phase angles α, β, γ and δ (and certainly so when the phase angle difference between all signals is constant), it can be observed that $\cos 4\eta > 0$ for those symbols with a 4th bit of a '0', and $\cos 4\eta < 0$ for those symbols with a 4th bit of a '1' assuming that there is no interference. Since it can be shown that [1]

$$a^4 \cos 4\eta = (r_1^2 - r_2^2)^2 - 4r_1^2 r_2^2$$

and $a^4 > 0$, decisions can be made based on whether $a^4 \cos 4\eta$ exceeds or is exceeded by zero, and are implemented based on whether $(r_1^2 - r_2^2)^2 - 4r_1^2 r_2^2 > 0$ or $(r_1^2 - r_2^2)^2 - 4r_1^2 r_2^2 < 0$.

In a similar form, the phase η can be used in the determination of the fifth bit by focusing on the sign of the term $a^8 \cos 8\eta$. That is, $\cos 8\eta > 0$ for symbols with a 5th bit of a '0' and $\cos 8\eta < 0$ for symbols with a 5th bit of a '1'. The implementation of this test in terms of r_1 and r_2 can be worked out as follows

$$\begin{aligned} a^8 \cos 8\eta &= a^8 \cos(4\eta + 4\eta) \\ &= a^8 \cos^2 4\eta - a^8 \sin^2 4\eta \\ &= (a^4 \cos 4\eta)^2 - (a^4 \sin 4\eta)^2 \end{aligned}$$

The term "$a^4 \cos 4\eta$" is available from the determination of the fourth bit. (Observe that $a^8 > 0$ and cannot influence the sign of $a^8 \cos 8\eta$). Voltages r_1 and r_2 are defined in terms of the angle η and 'a' as

$$r_1 = a \cos\eta \tag{3.3}$$

$$r_2 = a \sin\eta \tag{3.4}$$

It is clear that

$$\begin{aligned} a^4 \sin 4\eta &= a^4 (2 \sin 2\eta \cos 2\eta) \\ &= 2\, a^4 (2 \sin\eta \cos\eta \cos 2\eta) \\ &= 4\, a^4 \sin\eta \cos\eta (\cos^2\eta - \sin^2\eta) \\ &= 4\, (a \sin\eta)(a \cos\eta)(a^2 \cos^2\eta - a^2 \sin^2\eta) \\ &= 4\, r_1\, r_2\, (r_1^2 - r_1^2) \end{aligned}$$

so that

$$a^8 \cos 8\eta = [(r_1^2 - r_2^2)^2 - 4r_1^2 r_2^2]^2 - [4r_1 r_2 (r_1^2 - r_2^2)]^2$$

and simplifying this we obtain

$$a^8\cos 8\eta = (r_1{}^2 - r_2{}^2)^4 - 24\, r_1{}^2 r_2{}^2 (r_1{}^2 - r_2{}^2) + 16 r_1{}^4 r_2{}^4 \qquad (3.5)$$

This term must be implemented in hardware for the recovery of the 5th bit using the correlator outputs r_1 and r_2. However, noise performance analyses will be carried out by evaluating the probability that $\cos 8\eta \geq 0$.

The complete design of the DBD receiver is shown in Figure 3.2. Note that for the first four bits, the design is similar to that used for 16-PSK modulated signals with the hardware for the recovery of the fifth bit simply added to it, taking advantage of the computed terms already available in that design.

B. NOISE PERFORMANCE ANALYSIS

It has been stated in the previous chapter that the interference model to be used in the receiver performance analysis is the Additive White Gaussian Noise model.

Using the previously described properties of the AWGN and the rotational symmetry of signal vectors, under the assumption that all the signals are equiprobable, the bit error rate for each bit can be obtained by considering only the conditional BER assuming that the signals in the first quadrant only, that is, $s_i(t)$, $i = 1,2$ ----,8 are transmitted.

The transmitted signals, which are represented as vectors in Figure 3.1, can be mathematically expressed as

$$s_i(t) = \sqrt{2E_s/T_s}\,\sin[2\pi f_c t + \theta_i(t)] \qquad 0 \leq t \leq T_s \qquad i = 1,2,3,\text{----},32 \qquad (3.6)$$

where $\theta_i(t)$ is given by

$\alpha + (i-1)\pi/16$ for $i = 1,9,17,25$

$\beta + (i-2)\pi/16$ for $i = 2,10,18,26$

$\gamma + (i-3)\pi/16$ for $i = 3,11,19,27$

$\delta + (i-4)\pi/16$ for $i = 4,12,20,28$

$(i+3)\pi/16 - \delta$ for $i = 5,13,21,29$

$(i+2)\pi/16 - \gamma$ for $i = 6,14,22,30$

$(i+1)\pi/16 - \beta$ for $i = 7,15,23,31$

$(i)\pi/16 - \alpha$ for $i = 8,16,24,32$

As demonstrated by Equation 2.20, Equation 3.6 can be written as

$$s_i(t) = \sqrt{E_s}\,\cos\theta_i(t)\,\varphi_1(t) + \sqrt{E_s}\,\sin\theta_i(t)\,\varphi_2(t) \qquad 0 \leq t \leq T_s$$

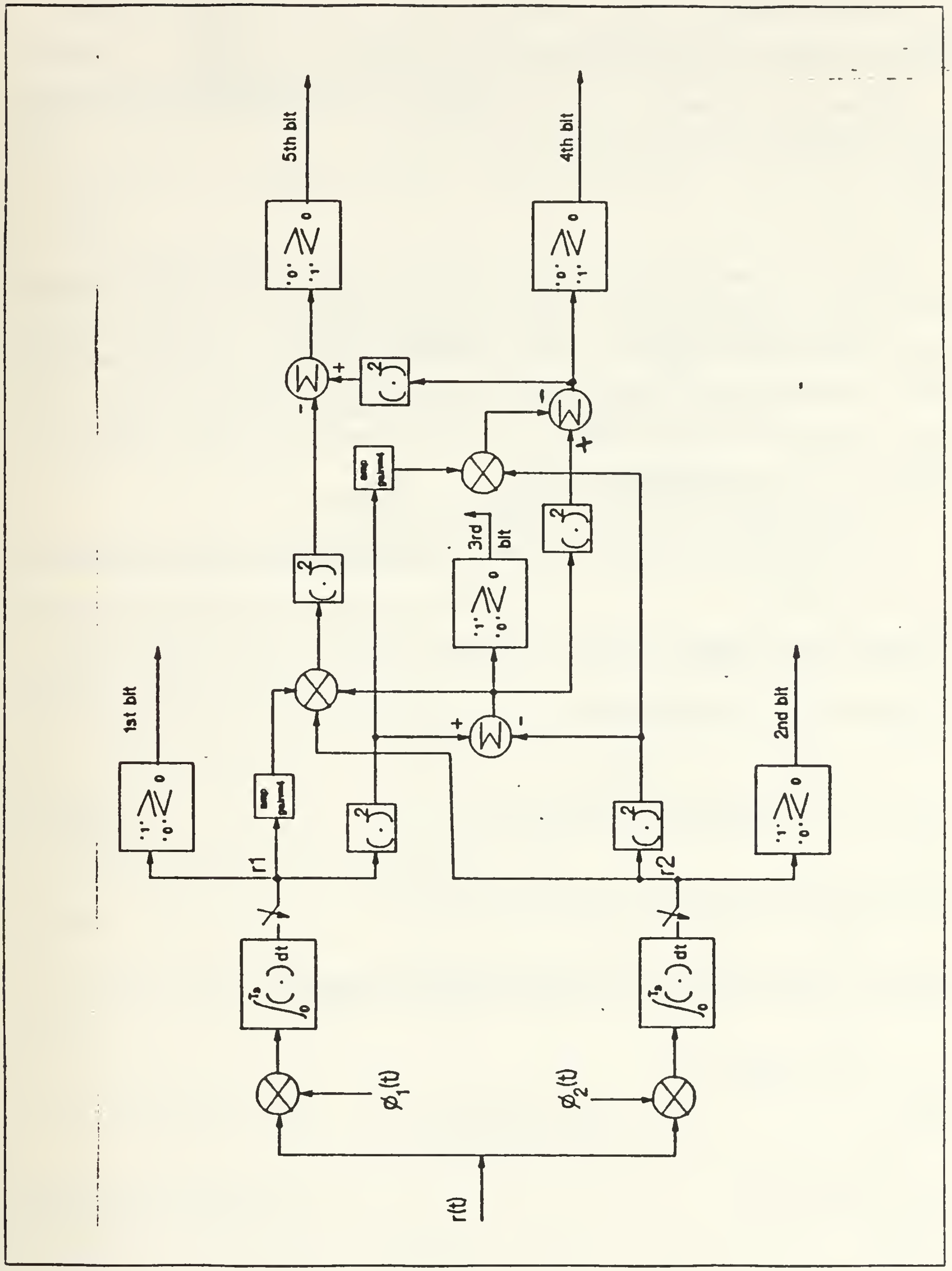

Figure 3.2 Proposed DBD Receiver Design for 32-PSK Modulated Signals.

$$i = 1,2,\text{----},32 \tag{3.7}$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are defined by Equations 2.18 and 2.19 respectively .

We assume that every T_s seconds the signal $r(t)$ is received which consists of the transmitted signal $s_i(t)$ and the interference $n(t)$ so that

$$r(t) = s_i(t) + n(t) \tag{3.8}$$

Here $s_i(t)$ is given by Equation 3.6 and $n(t)$ is a sample function of the Additive White Gaussian Noise (AWGN) of power spectral density level $N_o/2$ watts per Hz. Using the signal space diagram and the described methodology by which the DBD receiver recovers each bit, the noise performance analysis is carried out for each bit separately, with all the results combined in the end to produce the receiver's BER as a function of SNR and the signal phase angles.

1. Noise Analysis Of The First Bit (MSB)

If we assume that $s_1(t)$ is transmitted, then, from the signal space diagram, it can be observed that

$$\Pr\{\text{MSB correct} / s_1(t)\} = \Pr\{ r_1 > 0 / s_1(t) \} \tag{3.9}$$

Now, assuming $s_1(t)$ was transmitted,

$$r_1/s_1(t) = \int_0^{T_s} s_1(t)\,\varphi_1(t)\,dt + \int_0^{T_s} n(t)\,\varphi_1(t)\,dt \tag{3.10}$$

By using Equation 3.6, Equation 3.10 can be written as

$$r_1/s_1(t) = \int_0^{T_s} [\sqrt{E_s}\cos\theta_1(t)\varphi_1(t) + \sqrt{E_s}\sin\theta_1(t)\varphi_2(t) + n(t)]\varphi_1(t)\,dt$$

$$= \sqrt{E_s}\cos\alpha + n_1$$

where

$$n_1 = \int_0^{T_s} n(t)\,\varphi_1(t)\,dt \tag{3.11}$$

It can be observed that n_1 is a zero mean Gaussian random variable having variance of $N_0/2$ [8]. This implies that the probability density function (p.d.f) of $r_1/s_1(t)$ will also be Gaussian and is given by

$$f_{r_1/s_1(t)}(R_1/s_1) = \frac{1}{\sqrt{2\pi N_0/2}} \exp[-(R_1 - \sqrt{E_s}\cos\alpha)^2/2N_0/2\,] \qquad (3.12)$$

This means that

$$\Pr\{\, r_1 > 0/s_1(t)\,\} = \int_0^\infty \frac{1}{\sqrt{2\pi N_0/2}} \exp[-(R_1 - \sqrt{E_s}\cos\alpha)^2/2N_0/2\,]\, dR_1 \qquad (3.13)$$

By changing integration variables, this can be expressed as

$$\begin{aligned}\Pr\{\, r_1 > 0 / s_1(t)\,\} &= \int_{-\sqrt{2E_s/N_0}\cos\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-Z^2/2)\, dZ \qquad (3.14)\\ &= Q(-\sqrt{2E_s/N_0}\cos\alpha)\end{aligned}$$

by using the definition of Q(.) given in Equation 2.26.

Using the same approach, assuming the remaining seven signals in the first quadrant are transmitted, we can write

$$\Pr\{\text{MSB correct} / s_2(t)\} = Q(-\sqrt{2E_s/N_0}\cos\beta) \qquad (3.15)$$

and

$$\Pr\{\text{MSB correct} / s_3(t)\} = Q(-\sqrt{2E_s/N_0}\cos\gamma) \qquad (3.16)$$

as well as

$$\Pr\{\text{MSB correct} / s_4(t)\} = Q(-\sqrt{2E_s/N_0}\cos\delta) \qquad (3.17)$$

Similarly

$$\begin{aligned}\Pr\{\text{MSB correct} / s_5(t)\,\} &= Q(-\sqrt{2E_s/N_0}\cos(\pi/2 - \delta)) \qquad (3.18)\\ &= Q(-\sqrt{2E_s/N_0}\sin\delta)\end{aligned}$$

Using the same approach,

$$\Pr\{\text{MSB correct} / s_6(t)\} = Q(-\sqrt{2E_s/N_0}\sin\gamma) \qquad (3.19)$$

while

$$\Pr\{\text{MSB correct} / s_7(t)\} = Q(-\sqrt{2E_s/N_0}\sin\beta) \qquad (3.20)$$

and finally

$$\Pr\{\text{MSB correct} / s_8(t)\} = Q(-\sqrt{2E_s/N_o}\sin\alpha) \tag{3.21}$$

Because of rotational symmetry, which is evident from the signal structures and the bit-to-symbol mappings, it can be stated that

$$\Pr\{\text{MSB correct} / s_k(t)\} = \Pr\{\text{MSB correct} / s_i(t)\} \tag{3.22}$$

where for

$k = 1$, $i = 16,17,32$
$k = 2$, $i = 15,18,31$
$k = 3$, $i = 14,19,30$
$k = 4$, $i = 13,20,29$
$k = 5$, $i = 12,21,28$
$k = 6$, $i = 11,22,27$
$k = 7$, $i = 10,23,26$
$k = 8$, $i = 9,24,25$

Using now these results, a consolidated expression can be developed for the probability of the first bit being correctly recovered, assuming that all the signals are equally likely to be transmitted. This means that

$$\Pr\{\text{MSB correct}\} = \frac{1}{32}\sum_{i=1}^{32} \Pr\{\text{MSB correct}/s_i(t)\} \tag{3.23}$$

and from the equalities given by Equation 3.22, it is apparent that the unconditional probability of correct reception of the MSB is given by

$$\Pr\{\text{MSB correct}\} = \frac{1}{8}\sum_{i=1}^{8} \Pr\{\text{MSB correct}/s_i(t)\} \tag{3.24}$$

From Equation 3.14 - 3.21 and the fact that

$$\Pr\{\text{MSB in error}\} = 1 - \Pr\{\text{MSB correct}\} \tag{3.25}$$

as well as

$$Q(-K) = 1 - Q(K) \tag{3.26}$$

we obtain

$$\Pr\{\text{MSB in error}\} = 1\text{-}1/8[1\text{-}Q(\sqrt{2E_s/N_0}\cos\alpha)+1\text{-}Q(\sqrt{2E_s/N_0}\cos\beta)$$
$$+1\text{-}Q(\sqrt{2E_s/N_0}\cos\gamma)+1\text{-}Q(\sqrt{2E_s/N_0}\cos\delta)+1\text{-}Q(\sqrt{2E_s/N_0}\sin\alpha)$$
$$+1\text{-}Q(\sqrt{2E_s/N_0}\sin\beta)+1\text{-}Q(\sqrt{2E_s/N_0}\sin\gamma)+1\text{-}Q(\sqrt{2E_s/N_0}\sin\delta)] \qquad (3.27)$$

which can further be simplified to the final form

$$\Pr\{\text{MSB in error}\} = 1/8[Q(\sqrt{2E_s/N_0}\cos\alpha)+Q(\sqrt{2E_s/N_0}\cos\beta)$$
$$+Q(\sqrt{2E_s/N_0}\cos\gamma)+Q(\sqrt{2E_s/N_0}\cos\delta)+Q(\sqrt{2E_s/N_0}\sin\alpha)$$
$$+Q(\sqrt{2E_s/N_0}\sin\beta)+Q(\sqrt{2E_s/N_0}\sin\gamma)+Q(\sqrt{2E_s/N_0}\sin\delta)] \equiv \text{PE1} \qquad (3.28)$$

2. Noise Analysis Of The Second Bit (2nd bit)

If we assume that $s_1(t)$ is transmitted, then from the signal space diagram it can be observed that

$$\Pr\{\text{2nd bit correct} / s_1(t)\} = \Pr\{ r_2 > 0 / s_1(t) \} \qquad (3.29)$$

Since assuming that $s_1(t)$ is transmitted

$$r_2/s_1(t) = \int_0^{T_s} s_1(t)\,\varphi_2(t)\,dt + \int_0^{T_s} n(t)\,\varphi_2(t)\,dt \qquad (3.30)$$

By using Equation 3.7, Equation 3.30 can be written as

$$r_2(t)/s_1(t) = \int_0^{T_s} [\sqrt{E_s}\cos\theta_1(t)\varphi_1(t)+\sqrt{E_s}\sin\theta_1(t)\varphi_2(t)+n(t)]\varphi_2(t).\,dt$$
$$=\sqrt{E_s}\sin\alpha + n_2$$

where

$$n_2 = \int_0^{T_s} n(t)\,\varphi_2(t)\,dt \qquad (3.31)$$

It can be observed that n_2 is a zero mean Gaussian random variable having variance of $N_0/2$ [8]. It can be further demonstrated that

$$E\{n_1 n_2\} = 0 \qquad (3.32)$$

implying that n_1 and n_2 are uncorrelated. Since n_1 and n_2 are Gaussian random variables, they are also statistically independent. The probability density function (p.d.f) of $r_2/s_1(t)$ is given by

$$f_{r_2/s_1(t)}(R_2/s_1) = \frac{1}{\sqrt{2\pi N_o/2}} \exp[-(R_2 - \sqrt{E_s}\sin\alpha)^2/2N_o/2\,] \qquad (3.33)$$

This means that

$$\Pr\{\, r_2 > 0/s_1(t)\,\} = \int_0^\infty \frac{1}{\sqrt{2\pi N_o/2}} \exp[-(R_2 - \sqrt{E_s}\sin\alpha)^2/2N_o/2\,]\, dR_2$$
$$= Q(-\sqrt{2E_s/N_o}\,\sin\alpha) = \Pr\{\text{2nd bit correct}/s_1(t)\} \qquad (3.34)$$

Using a similar approach, when conditioned upon $s_i(t)$, $i=2,3,\text{-----},8$, the probability of the 2nd bit being correct is given by

$$\Pr\{\text{2nd bit correct} / s_2(t)\} = Q(-\sqrt{2E_s/N_o}\,\sin\beta) \qquad (3.35)$$

and

$$\Pr\{\text{2nd bit correct} / s_3(t)\} = Q(-\sqrt{2E_s/N_o}\,\sin\gamma) \qquad (3.36)$$

as well as

$$\Pr\{\text{2nd bit correct} / s_4(t)\} = Q(-\sqrt{2E_s/N_o}\,\sin\delta) \qquad (3.37)$$

Similarly

$$\Pr\{\text{2nd bit correct} / s_5(t)\,\} = Q(-\sqrt{2E_s/N_o}\,\sin(\pi/2 - \delta)) \qquad (3.38)$$
$$= Q(-\sqrt{2E_s/N_o}\,\cos\delta)$$

and

$$\Pr\{\text{2nd bit correct} / s_6(t)\} = Q(-\sqrt{2E_s/N_o}\,\cos\gamma) \qquad (3.39)$$

Also

$$\Pr\{\text{2nd bit correct} / s_7(t)\} = Q(-\sqrt{2E_s/N_o}\,\cos\beta) \qquad (3.40)$$

and finally

$$\Pr\{\text{2nd bit correct} / s_8(t)\} = Q(-\sqrt{2E_s/N_o}\,\cos\alpha) \qquad (3.41)$$

The equalities given by Equation 3.22 hold as well for the second bit, so that the unconditional probability of correct reception of the 2nd bit is given by

$$\Pr\{\text{2nd bit correct}\} = \frac{1}{8}\sum_{i=1}^{8} \Pr\{\text{2nd bit correct}/s_i(t)\} \tag{3.42}$$

From Equations 3.34 - 3.41, it is possible to obtain

$$\begin{aligned}\Pr\{\text{2nd bit in error}\} = {} & 1-1/8[1-Q(\sqrt{2E_s/N_0}\sin\alpha)+1-Q(\sqrt{2E_s/N_0}\sin\beta) \\ & +1-Q(\sqrt{2E_s/N_0}\sin\gamma)+1-Q(\sqrt{2E_s/N_0}\sin\delta)+1-Q(\sqrt{2E_s/N_0}\cos\alpha) \\ & +1-Q(\sqrt{2E_s/N_0}\cos\beta)+1-Q(\sqrt{2E_s/N_0}\cos\gamma)+1-Q(\sqrt{2E_s/N_0}\cos\delta)]\end{aligned} \tag{3.43}$$

which can further be simplified to the final form

$$\begin{aligned}\Pr\{\text{2nd bit in error}\} = {} & 1/8[Q(\sqrt{2E_s/N_0}\sin\alpha)+Q(\sqrt{2E_s/N_0}\sin\beta) \\ & +Q(\sqrt{2E_s/N_0}\sin\gamma)+Q(\sqrt{2E_s/N_0}\sin\delta)+Q(\sqrt{2E_s/N_0}\cos\alpha) \\ & +Q(\sqrt{2E_s/N_0}\cos\beta)+Q(\sqrt{2E_s/N_0}\cos\gamma)+Q(\sqrt{2E_s/N_0}\cos\delta)] \equiv \text{PE2}\end{aligned} \tag{3.44}$$

Examination of the expressions for PE1 and PE2 reveals that PE1 = PE2 as expected.

3. Noise Analysis Of The Third Bit (3rd bit)

It has been explained while considering the receiver design that for the third bit to be correctly recovered, conditioned on $s_i(t)$, i = 1,2,3,4 being transmitted

$$(r_1^2 - r_2^2) > 0 \tag{3.45}$$

must hold, while

$$(r_1^2 - r_2^2) < 0 \tag{3.46}$$

when $s_i(t)$, i = 5,6,7,8 is transmitted,

Assume, therefore, that $s_1(t)$ has been transmitted so that

$$\begin{aligned}\Pr\{\text{3rd bit correct} / s_1(t)\} &= \Pr\{(r_1^2 - r_2^2) > 0 / s_1(t)\} \\ &= \Pr\{(r_1+r_2)(r_1-r_2) > 0 / s_1(t)\}\end{aligned} \tag{3.47}$$

Then under the condition that $s_1(t)$ is transmitted, define

$$X = r_1 + r_2 \quad \text{and} \quad Y = r_1 - r_2 \tag{3.48}$$

Using the results of Equations 3.10 and 3.30, we obtain

$$X/s_1(t) = \sqrt{E_s}\cos\alpha + n_1 + \sqrt{E_s}\sin\alpha + n_2 \tag{3.49}$$

so that

$$E\{X/s_1(t)\} = \sqrt{E_s}(\cos\alpha + \sin\alpha) \tag{3.50}$$

and

$$\mathrm{var}\{X/s_1(t)\} = E\{(n_1+n_2)^2\} = N_o \tag{3.51}$$

since, as previously demonstrated,

$E\{n_1n_2\} = 0$ and $E\{n_1{}^2\} = E\{n_2{}^2\} = N_o/2$

Similarly

$$Y/s_1(t) = \sqrt{E_s}\cos\alpha + n_1 - \sqrt{E_s}\sin\alpha - n_2 \tag{3.52}$$

so that

$$E\{Y/s_1(t)\} = \sqrt{E_s}(\cos\alpha - \sin\alpha) \tag{3.53}$$

and

$$\mathrm{var}\{Y/s_1(t)\} = E\{(n_1-n_2)^2\} = N_o \tag{3.54}$$

Furthermore,

$E\{[X-E\{X/s_1(t)\}][Y-E\{Y/s_1(t)\}]\} = E\{(n_1+n_2)(n_1-n_2)\} = 0$

so that X and Y are conditionally independent random variables, making it possible to write

$$\begin{aligned}&\Pr\{(r_1+r_2)(r_1-r_2) > 0 / s_1(t)\} = \Pr\{XY > 0 / s_1(t)\}\\&= \Pr\{X > 0 / s_1(t)\}\Pr\{Y > 0 / s_1(t)\} + \Pr\{X < 0 / s_1(t)\}\Pr\{Y < 0 / s_1(t)\}\end{aligned} \tag{3.55}$$

These kind of expressions have already been dealt with in previous sections so that using available results and the fact that

$\Pr\{X < 0 / s_1(t)\} = 1 - \Pr\{X > 0 / s_1(t)\}$

the following expression can be obtained

$$\begin{aligned}&\Pr\{\text{3rd bit correct} / s_1(t)\} = Q(-\sqrt{E_s/N_o}(\cos\alpha+\sin\alpha))Q(-\sqrt{E_s/N_o}(\cos\alpha-\sin\alpha))\\&+\{1-Q(-\sqrt{E_s/N_o}(\cos\alpha+\sin\alpha))\}\{1-Q(-\sqrt{E_s/N_o}(\cos\alpha-\sin\alpha))\}\end{aligned} \tag{3.56}$$

Using an analogous approach, similar expressions can be developed for the probability of the third bit being correct conditioned on $s_2(t)$, $s_3(t)$ and $s_4(t)$ being transmitted so that

$$\begin{aligned}&\Pr\{\text{3rd bit correct} / s_2(t)\} = Q(-\sqrt{E_s/N_o}(\cos\beta+\sin\beta))Q(-\sqrt{E_s/N_o}(\cos\beta-\sin\beta))\\&+\{1-Q(-\sqrt{E_s/N_o}(\cos\beta+\sin\beta))\}\{1-Q(-\sqrt{E_s/N_o}(\cos\beta-\sin\beta))\}\end{aligned} \tag{3.57}$$

Similarly,

$$\Pr\{\text{3rd bit correct} / s_3(t)\} = Q(-\sqrt{E_s/N_0}(\cos\gamma + \sin\gamma))Q(-\sqrt{E_s/N_0}(\cos\gamma - \sin\gamma)) + \{1-Q(-\sqrt{E_s/N_0}(\cos\gamma + \sin\gamma))\}\{1-Q(-\sqrt{E_s/N_0}(\cos\gamma - \sin\gamma))\} \quad (3.58)$$

and

$$\Pr\{\text{3rd bit correct} / s_4(t)\} = Q(-\sqrt{E_s/N_0}(\cos\delta + \sin\delta))Q(-\sqrt{E_s/N_0}(\cos\delta - \sin\delta)) + \{1-Q(-\sqrt{E_s/N_0}(\cos\delta + \sin\delta))\}\{1-Q(-\sqrt{E_s/N_0}(\cos\delta - \sin\delta))\} \quad (3.59)$$

Now, for the probability of the third bit being correctly recovered, conditioned on $s_i(t)$, $i = 5,6,7,8$ being transmitted, from Equation 3.46 it can be seen that

$$\begin{aligned} \Pr\{\text{3rd bit correct} / s_5(t)\} &= \Pr\{(r_1^2 - r_2^2) < 0 / s_5(t)\} \quad (3.60) \\ &= 1 - \Pr\{(r_1^2 - r_2^2) > 0 / s_5(t)\} \end{aligned}$$

Since expressions for the above probability have already been worked out previously, it is possible to write directly

$$\Pr\{\text{3rd bit correct} / s_5(t)\} = 1 - [Q(-\sqrt{E_s/N_0}(\cos(\pi/2-\delta) + \sin(\pi/2-\delta))) Q(-\sqrt{E_s/N_0}(\cos(\pi/2-\delta) - \sin(\pi/2-\delta))) + \{1-Q(-\sqrt{E_s/N_0}(\cos(\pi/2-\delta) + \sin(\pi/2-\delta)))\} \{1-Q(-\sqrt{E_s/N_0}(\cos(\pi/2-\delta) - \sin(\pi/2-\delta)))\}] \quad (3.61)$$

which when simplified can be written as

$$\Pr\{\text{3rd bit correct} / s_5(t)\} = 1 - Q(-\sqrt{E_s/N_0}(\sin\delta + \cos\delta))Q(-\sqrt{E_s/N_0}(\sin\delta - \cos\delta)) - \{1-Q(-\sqrt{E_s/N_0}(\sin\delta + \cos\delta))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\delta - \cos\delta))\} \quad (3.62)$$

Using a similar procedure, we obtain

$$\Pr\{\text{3rd bit correct} / s_6(t)\} = 1 - Q(-\sqrt{E_s/N_0}(\sin\gamma + \cos\gamma))Q(-\sqrt{E_s/N_0}(\sin\gamma - \cos\gamma)) - \{1-Q(-\sqrt{E_s/N_0}(\sin\gamma + \cos\gamma))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\gamma - \cos\gamma))\} \quad (3.63)$$

as well as

$$\Pr\{\text{3rd bit correct} / s_7(t)\} = 1 - Q(-\sqrt{E_s/N_0}(\sin\beta + \cos\beta)Q(-\sqrt{E_s/N_0}(\sin\beta - \cos\beta) - \{1-Q(-\sqrt{E_s/N_0}(\sin\beta + \cos\beta))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\beta - \cos\beta))\} \quad (3.64)$$

and finally

$$\Pr\{\text{3rd bit correct} / s_8(t)\} = 1 - Q(-\sqrt{E_s/N_0}(\sin\alpha + \cos\alpha)Q(-\sqrt{E_s/N_0}(\sin\alpha - \cos\alpha) - \{1-Q(-\sqrt{E_s/N_0}(\sin\alpha + \cos\alpha))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\alpha - \cos\alpha))\} \quad (3.65)$$

The equalities stated in Equation 3.22 hold true for the evaluations involving the third bit also, so that as previously carried out, assuming all signals are equally likely to be transmitted, the unconditional probability of the third bit being correct is obtained from

$$\Pr\{\text{3rd bit correct}\} = \frac{1}{8}\sum_{i=1}^{8} \Pr\{\text{3rd bit correct}/s_i(t)\} \qquad (3.66)$$

so that Equation 3.66 can be written as

$$\begin{aligned}
\Pr\{\text{3rd bit in error}\} = {} & 1-1/8\,[Q(-\sqrt{E_s/N_0}(\cos\alpha+\sin\alpha))Q(-\sqrt{E_s/N_0}(\cos\alpha-\sin\alpha)) \\
& +\{1-Q(-\sqrt{E_s/N_0}(\cos\alpha+\sin\alpha))\}\{1-Q(-\sqrt{E_s/N_0}(\cos\alpha-\sin\alpha))\} \\
& +Q(-\sqrt{E_s/N_0}(\cos\beta+\sin\beta))Q(-\sqrt{E_s/N_0}(\cos\beta-\sin\beta)) \\
& +\{1-Q(-\sqrt{E_s/N_0}(\cos\beta+\sin\beta))\}\{1-Q(-\sqrt{E_s/N_0}(\cos\beta-\sin\beta))\} \\
& +Q(-\sqrt{E_s/N_0}(\cos\gamma+\sin\gamma))Q(-\sqrt{E_s/N_0}(\cos\gamma-\sin\gamma)) \\
& +\{1-Q(-\sqrt{E_s/N_0}(\cos\gamma+\sin\gamma))\}\{1-Q(-\sqrt{E_s/N_0}(\cos\gamma-\sin\gamma))\} \\
& +Q(-\sqrt{E_s/N_0}(\cos\delta+\sin\delta))Q(-\sqrt{E_s/N_0}(\cos\delta-\sin\delta)) \\
& +\{1-Q(-\sqrt{E_s/N_0}(\cos\delta+\sin\delta))\}\{1-Q(-\sqrt{E_s/N_0}(\cos\delta-\sin\delta))\} \\
& +1-Q(-\sqrt{E_s/N_0}(\sin\delta+\cos\delta))+Q(-\sqrt{E_s/N_0}(\sin\delta-\cos\delta)) \\
& -\{1-Q(-\sqrt{E_s/N_0}(\sin\delta+\cos\delta))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\delta-\cos\delta))\} \\
& +1-Q(-\sqrt{E_s/N_0}(\sin\gamma+\cos\gamma))Q(-\sqrt{E_s/N_0}(\sin\gamma-\cos\gamma)) \\
& -\{1-Q(-\sqrt{E_s/N_0}(\sin\gamma+\cos\gamma))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\gamma-\cos\gamma))\} \\
& +1-Q(-\sqrt{E_s/N_0}(\sin\beta+\cos\beta))Q(-\sqrt{E_s/N_0}(\sin\beta-\cos\beta)) \\
& -\{1-Q(-\sqrt{E_s/N_0}(\sin\beta+\cos\beta))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\beta-\cos\beta))\} \\
& +1-Q(-\sqrt{E_s/N_0}(\sin\alpha+\cos\alpha))Q(-\sqrt{E_s/N_0}(\sin\alpha-\cos\alpha)) \\
& -\{1-Q(-\sqrt{E_s/N_0}(\sin\alpha+\cos\alpha))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\alpha-\cos\alpha))\}] \qquad (3.67)
\end{aligned}$$

which after collecting similar terms and simplifying takes the final form as

$$\begin{aligned}
\Pr\{\text{3rd bit in error}\} = {} & 1/4\,Q(\sqrt{E_s/N_0}(\cos\alpha-\sin\alpha)) + Q(\sqrt{E_s/N_0}(\cos\alpha+\sin\alpha)) \\
& \{1/4 - 1/2\,Q(\sqrt{E_s/N_0}(\cos\alpha-\sin\alpha))\} + 1/4\,Q(\sqrt{E_s/N_0}(\cos\beta-\sin\beta)) \\
& +Q(\sqrt{E_s/N_0}(\cos\beta+\sin\beta))\{1/4 - 1/2\,Q(\sqrt{E_s/N_0}(\cos\beta-\sin\beta)))\} \\
& +1/4\,Q(\sqrt{E_s/N_0}(\cos\gamma-\sin\gamma)) + Q(\sqrt{E_s/N_0}(\cos\gamma+\sin\gamma)) \\
& \{1/4 - 1/2\,Q(\sqrt{E_s/N_0}(\cos\gamma-\sin\gamma))\} + 1/4\,Q(\sqrt{E_s/N_0}(\cos\delta-\sin\delta)) \\
& +Q(\sqrt{E_s/N_0}(\cos\delta+\sin\delta))\{1/4 - 1/2\,Q(\sqrt{E_s/N_0}(\cos\delta-\sin\delta))\} \equiv PE3 \qquad (3.68)
\end{aligned}$$

whether $(r_1^2 - r_2^2) - 4 r_1^2 r_2^2$ exceeds or is exceeded by zero. Therefore, assuming that $s_1(t)$ is transmitted

$$\Pr\{\text{4th bit correct} / s_1(t)\} = \Pr\{(r_1^2 - r_2^2)^2 - 4r_1^2 r_2^2 > 0 / s_1(t)\} \tag{3.69}$$

It appears that, even though r_1 and r_2 are Gaussian random variables, obtaining the above probability is nearly impossible as the transformation of r_1 and r_2 involves a very non-linear function that does not lend itself to p.d.f determination .

A simpler approach can be adopted by using Equations 3.1 - 3.4 in which by substitution and use of trignometeric identities it is possible to express [see ref. 2] :

$$(r_1^2 - r_2^2)^2 - 4r_1^2 r_2^2 = a^4 \cos 4\eta \tag{3.70}$$

Therefore the probability of making a correct decision on the 4th bit can be obtained by computing (assuming $s_1(t)$ is transmitted)

$$\begin{aligned}\Pr\{\text{4th bit correct} / s_1(t)\} &= \Pr\{a^4 \cos 4\eta > 0 / s_1(t)\} \\ &= \Pr\{\cos 4\eta > 0 / s_1(t)\}\end{aligned} \tag{3.71}$$

This indicates that the above probability can be obtained by finding the p.d.f of the phase 'η'-conditioned on $s_i(t)$, for i = 1,2,-----,32

Assume for example that $s_1(t)$ has been transmitted so that from previous work, conditioned on $s_1(t)$ being transmitted,

$$r_1 \sim N(\sqrt{E_s}\cos\alpha\, ,\, N_o/2)$$

and

$$r_2 \sim N(\sqrt{E_s}\sin\alpha\, ,\, N_o/2)$$

where N(m,v) implies a Gaussian p.d.f of mean m and variance v. Furthermore, it has been demonstrated that r_1 and r_2 are statistically independent Gaussian random variables so that, by using Equation 3.12 and 3.33, the joint p.d.f of r_1 and r_2 conditioned on $s_1(t)$ being transmitted can be written as

$$f(R_1,R_2/s_1) = \frac{1}{\sqrt{2\pi N_o/2}} \exp[-(R_1 - \sqrt{E_s}\cos\alpha)^2/2N_o/2\,]\, \frac{1}{\sqrt{2\pi N_o/2}} \exp[-(R_2 - \sqrt{E_s}\sin\alpha)^2/2N_o/2\,] \tag{3.72}$$

Using random variable transformation techniques, the joint p.d.f of 'a' and η can be given by

$$f_{a,\eta/s_1(t)}(A,H) = |A| f_{r_1 r_2/s_1(t)}(A\cos H, A\sin H) + |A| f_{r_1 r_2/s_1(t)}(-A\cos H, -A\sin H)$$
$$0 \leq H \leq 2\pi \qquad (3.73)$$

Equation 3.73 can be re-expressed by using Equation 3.72, with the final expression after integrating out the r.v. 'a' and simplifying, written as [see ref. 2]

$$f_{\eta/s_1(t)}(H/s_1) = 1/2\pi \exp(-E_s/N_o)[1+\sqrt{2\pi E_s/N_o/2}\exp(E_s/N_o\cos^2(H-\alpha))$$
$$\cos(H-\alpha)\, Q(-\sqrt{2E_s/N_o}\cos(H-\alpha))] \qquad 0 \leq H \leq 2\pi \qquad (3.74)$$

Now, using the above p.d.f of η conditioned on $s_1(t)$ being transmitted and the fact that

$$\cos 4\eta > 0, \quad \text{for } -\pi/8 \leq \eta \leq \pi/8,\ 3\pi/8 \leq \eta \leq 5\pi/8$$
$$7\pi/8 \leq \eta \leq 9\pi/8 \text{ and } 11\pi/8 \leq \eta \leq 13\pi/8$$

then

$$\Pr\{\text{4th bit correct}/s_1(t)\} = \Pr\{\cos 4\eta > 0/s_1(t)\} \qquad (3.75)$$

$$= \int_{-\pi/8}^{\pi/8} f_{\eta/s_1(t)}(H/s_1)dH + \int_{3\pi/8}^{5\pi/8} f_{\eta/s_1(t)}(H/s_1)dH + \int_{7\pi/8}^{9\pi/8} f_{\eta/s_1(t)}(H/s_1)dH + \int_{11\pi/8}^{13\pi/8} f_{\eta/s_1(t)}(H/s_1)dH$$

The integrals can be modified by a change of variables to yield

$$= \int_{-\pi/8}^{\pi/8} [\, f_{\eta/s_1(t)}(H/s_1) + f_{\eta/s_1(t)}(H+\pi/2/s_1) + f_{\eta/s_1(t)}(H+\pi/s_1)$$
$$+ f_{\eta/s_1(t)}(H+3\pi/2/s_1)\,]\, dH \qquad (3.76)$$

The individual terms of Equation 3.76 can be obtained by using Equation 3.73. Namely

$$f_{\eta/s_1(t)}(H+\pi/2/s_1) = 1/2\pi \exp(-E_s/N_o)[1-\sqrt{2\pi E_s/N_o/2}\exp(E_s/N_o\sin^2(H-\alpha))$$
$$\sin(H-\alpha)\, Q(-\sqrt{2E_s/N_o}\sin(H-\alpha))] \qquad (3.77)$$

$$f_{\eta/s_1(t)}(H+\pi/s_1) = 1/2\pi \exp(-E_s/N_0)[1-\sqrt{2\pi E_s/N_0/2}\ \exp(E_s/N_0\cos^2(H-\alpha))\ \cos(H-\alpha)\ Q(-\sqrt{2E_s/N_0}\cos(H-\alpha))] \tag{3.78}$$

$$f_{\eta/s_1(t)}(H+3\pi/2/s_1) = 1/2\pi \exp(-E_s/N_0)[1+\sqrt{2\pi E_s/N_0/2}\ \exp(E_s/N_0\sin^2(H-\alpha))\ \sin(H-\alpha)\ Q(-\sqrt{2E_s/N_0}\sin(H-\alpha))] \tag{3.79}$$

Substituting these expressions in Equation 3.76 and simplifying by adding the appropriate terms, results in

$$\Pr\{4\text{th bit correct}/s_1(t)\} = 1/2\ \exp(-E_s/N_0)+\sqrt{E_s/2\pi N_0/2}\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_0\sin^2(H-\alpha))\ \cos(H-\alpha)\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha))-Q(\sqrt{2E_s/N_0}\cos(H-\alpha))\} + \exp(-E_s/N_0\ \cos^2(H-\alpha))\ \sin(H-\alpha)\{Q(-\sqrt{2E_s/N_0}\ \sin(H-\alpha)) - Q(\sqrt{2E_s/N_0}\ \sin(H-\alpha))\}]\ dH \tag{3.80}$$

Now, consider the probability of the fourth bit correctly recovered conditioned on $s_2(t)$ being transmitted. From the signal space diagram and the above explained logic,

$$\Pr\{4\text{th bit correct} / s_2(t)\} = \Pr\{\cos 4\eta > 0 / s_2(t)\} \tag{3.81}$$

From previous work, we know that conditioned on $s_2(t)$ being transmitted,

$$r_1 \sim N(\sqrt{E_s}\cos\beta\ ,\ N_0/2)$$

and

$$r_2 \sim N(\sqrt{E_s}\sin\beta\ ,\ N_0/2)$$

so that the p.d.f of η conditioned on $s_2(t)$ being transmitted is given by

$$f_{\eta/s_2(t)}(H/s_2) = 1/2\pi \exp(-E_s/N_0)[1+\sqrt{2\pi E_s/N_0/2}\ \exp(E_s/N_0\cos^2(H-\beta))\ \cos(H-\beta)\ Q(-\sqrt{2E_s/N_0}\cos(H-\beta))] \qquad 0\le H\le 2\pi \tag{3.82}$$

Using the same approach as used in the previous case, we can write

$$\Pr\{4\text{th bit correct}/s_2(t)\} = 1/2\ \exp(-E_s/N_0)+\sqrt{E_s/2\pi N_0/2}\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_0\sin^2(H-\beta))\ \cos(H-\alpha)\{Q(-\sqrt{2E_s/N_0}\cos(H-\beta))-Q(\sqrt{2E_s/N_0}\cos(H-\beta))\} + \exp(-E_s/N_0\ \cos^2(H-\beta)\ \sin(H-\beta)\{Q(-\sqrt{2E_s/N_0}\ \sin(H-\beta)) - Q(\sqrt{2E_s/N_0}\ \sin(H-\beta))\}]\ dH \tag{3.83}$$

Assume now that $s_3(t)$ is transmitted . The previously discussed decision rule is such that the probability of a correct decision becomes

$$\begin{aligned}\Pr\{\text{4th bit correct} / s_3(t)\} &= \Pr\{\cos 4\eta < 0 / s_3(t)\} \\ &= 1 - \Pr\{\cos 4\eta > 0 / s_3(t)\}\end{aligned} \tag{3.84}$$

Since these kind of expressions have already been dealt with, and the p.d.f $f_{\eta/s_3(t)}(H/s_3)$ can be obtained by a procedure to that previously carried out, we can directly write

$$\begin{aligned}&\Pr\{\text{4th bit correct}/s_3(t)\} = 1-1/2\exp(-E_s/N_0)-\sqrt{E_s/2\pi N_0}/2\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_0\sin^2(H-\gamma)) \\ &\cos(H-\alpha)\{Q(-\sqrt{2E_s/N_0}\cos(H-\gamma))-Q(\sqrt{2E_s/N_0}\cos(H-\gamma))\} + \exp(-E_s/N_0\cos^2(H-\gamma)) \\ &\sin(H-\gamma)\{Q(-\sqrt{2E_s/N_0}\sin(H-\gamma)) - Q(\sqrt{2E_s/N_0}\sin(H-\gamma))\}]\,dH\end{aligned} \tag{3.85}$$

and finally conditioned on $s_4(t)$ being transmitted, we obtain

$$\begin{aligned}&\Pr\{\text{4th bit correct} / s_4(t)\} = \Pr\{\cos 4\eta < 0 / s_4(t)\} \\ &= 1 - \Pr\{\cos 4\eta > 0 / s_4(t)\} = 1 - 1/2\exp(-E_s/N_0) \\ &-\sqrt{E_s/2\pi N_0}/2\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_0\sin^2(H-\delta))\cos(H-\alpha)\{Q(-\sqrt{2E_s/N_0}\cos(H-\delta)) \\ &-Q(\sqrt{2E_s/N_0}\cos(H-\delta))\} + \exp(-E_s/N_0\cos^2(H-\delta)) \\ &\sin(H-\delta)\{Q(-\sqrt{2E_s/N_0}\sin(H-\delta)) - Q(\sqrt{2E_s/N_0}\sin(H-\delta))\}]\,dH\end{aligned} \tag{3.86}$$

Similarly, without repeating the intermediate steps necessary to the derivations, which are however procedurally identical to the steps followed above, we have when $s_5(t)$ is transmitted

$$\begin{aligned}&\Pr\{\text{4th bit correct} / s_5(t)\} = \Pr\{\cos 4\eta < 0 / s_5(t)\} \\ &= 1 - \Pr\{\cos 4\eta > 0 / s_5(t)\} = 1 - 1/2\exp(-E_s/N_0) \\ &-\sqrt{E_s/2\pi N_0}/2\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_0\sin^2(H-(\pi/2-\delta))\cos(H-(\pi/2-\alpha) \\ &\{Q(-\sqrt{2E_s/N_0}\cos(H-(\pi/2-\delta)))-Q(\sqrt{2E_s/N_0}\cos(H-(\pi/2-\delta)))\} \\ &+\exp(-E_s/N_0\cos^2(H-(\pi/2-\delta)))\sin(H-(\pi/2-\delta)) \\ &\{Q(-\sqrt{2E_s/N_0}\sin(H-(\pi/2-\delta))) - Q(\sqrt{2E_s/N_0}\sin(H-(\pi/2-\delta)))\}]\,dH\end{aligned} \tag{3.87}$$

which after simplifying becomes

$$\Pr\{\text{4th bit correct}/s_5(t)\} = 1 - 1/2\exp(-E_s/N_o)$$

$$-\sqrt{E_s/2\pi N_o/2}\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_o\cos^2(H+\delta)\sin(H+\delta)$$

$$\{Q(-\sqrt{2E_s/N_o}\sin(H+\delta))-Q(\sqrt{2E_s/N_o}\sin(H+\delta))\} + \exp(-E_s/N_o\cos^2(H+\delta)$$

$$\cos(H+\delta)\{Q(-\sqrt{2E_s/N_o}\cos(H+\delta)) - Q(\sqrt{2E_s/N_o}\cos(H+\delta))\}]\,dH. \tag{3.88}$$

When $s_6(t)$ is assumed transmitted

$$\Pr\{\text{4th bit correct} / s_6(t)\} = \Pr\{\cos 4\eta < 0 / s_6(t)\}$$

$$= 1 - \Pr\{\cos 4\eta > 0 / s_6(t)\} = 1 - 1/2\exp(-E_s/N_o)$$

$$-\sqrt{E_s/2\pi N_o}/2\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_o\cos^2(H+\gamma)\sin(H+\gamma)$$

$$\{Q(-\sqrt{2E_s/N_o}\sin(H+\gamma))-Q(\sqrt{2E_s/N_o}\sin(H+\gamma))\} + \exp(-E_s/N_o\cos^2(H+\gamma)$$

$$\cos(H+\gamma)\{Q(-\sqrt{2E_s/N_o}\cos(H+\gamma)) - Q(\sqrt{2E_s/N_o}\cos(H+\gamma))\}]\,dH \tag{3.89}$$

For $s_7(t)$ assumed transmitted

$$\Pr\{\text{4th bit correct} / s_7(t)\} = \Pr\{\cos 4\eta > 0 / s_7(t)\}$$

$$= 1/2\exp(-E_s/N_o) + \sqrt{E_s/2\pi N_o/2}\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_o\cos^2(H+\beta))$$

$$\sin(H+\beta)\{Q(-\sqrt{2E_s/N_o}\sin(H+\beta))-Q(\sqrt{2E_s/N_o}\sin(H+\beta))\}$$

$$+\exp(-E_s/N_o\sin^2(H+\beta))\cos(H+\beta)$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H+\beta)) - Q(\sqrt{2E_s/N_o}\cos(H+\beta))\}]\,dH \tag{3.90}$$

Finally, conditioned on $s_8(t)$ being the transmitted signal,

$$\Pr\{\text{4th bit correct} / s_8(t)\} = \Pr\{\cos 4\eta > 0 / s_8(t)\}$$

$$= 1/2\exp(-E_s/N_o) + \sqrt{E_s/2\pi N_o/2}\int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_o\cos^2(H+\alpha))$$

$$\sin(H+\alpha)\{Q(-\sqrt{2E_s/N_o}\sin(H+\alpha))-Q(\sqrt{2E_s/N_o}\sin(H+\alpha))\}$$

$$+\exp(-E_s/N_o\cos^2(H+\alpha)\cos(H+\alpha)$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H+\alpha)) - Q(\sqrt{2E_s/N_o}\cos(H+\alpha))\}]\,dH \tag{3.92}$$

The equalities stated in the Equation 3.22 hold true for the 4th bit as well, so as previously carried out, these results are now used to develop a general expression for the probability of the 4th bit being correctly recovered, in which Equations 3.80 - 3.91 are used along with the assumption that all signals are equally likely to be transmitted, to yield

$$\begin{aligned} \Pr\{\text{4th bit correct}\} &= \frac{1}{8}\sum_{i=1}^{8} \Pr\{\text{4th bit correct}/s_i(t)\} \qquad (3.93) \\ &= 1 - \Pr\{\text{4th bit in error}\} \end{aligned}$$

so that

$$\begin{aligned} &\Pr\{\text{4th bit in error}\} = 1/2 - \sqrt{E_s/\pi N_0} \int_0^{\pi/8} [\, \exp(-E_s/N_0 \sin^2(H-\alpha)) \\ &\cos(H-\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H-\alpha))\} \\ &+\exp(-E_s/N_0 \cos^2(H-\alpha)) \sin(H-\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H-\alpha))\} \\ &+\exp(-E_s/N_0 \sin^2(H+\alpha)) \cos(H+\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H+\alpha))\} \\ &+\exp(-E_s/N_0 \cos^2(H+\alpha)) \sin(H+\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H+\alpha)) \\ &+\exp(-E_s/N_0 \sin^2(H-\beta)) \cos(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H-\beta))\} \\ &+\exp(-E_s/N_0 \cos^2(H-\beta)) \sin(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H-\beta))\} \\ &+\exp(-E_s/N_0 \sin^2(H+\beta)) \cos(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H+\beta))\} \\ &+\exp(-E_s/N_0 \cos^2(H+\beta)) \sin(H+\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H+\beta))\} \\ &-\exp(-E_s/N_0 \sin^2(H-\gamma)) \cos(H-\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H-\gamma))\} \\ &-\exp(-E_s/N_0 \cos^2(H-\gamma)) \sin(H-\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H-\gamma))\} \\ &-\exp(-E_s/N_0 \sin^2(H+\gamma)) \cos(H+\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H+\gamma))\} \\ &-\exp(-E_s/N_0 \cos^2(H+\gamma)) \sin(H+\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H+\gamma))\} \\ &-\exp(-E_s/N_0 \sin^2(H-\delta)) \cos(H-\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H-\delta))\} \\ &-\exp(-E_s/N_0 \cos^2(H-\delta)) \sin(H-\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H-\delta))\} \\ &-\exp(-E_s/N_0 \sin^2(H+\delta)) \cos(H+\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H+\delta))\} \\ &-\exp(-E_s/N_0 \cos^2(H+\delta)) \sin(H+\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H+\delta))\} \,]\, dH \\ &\quad \equiv PE4 \qquad (3.94) \end{aligned}$$

5. Noise Analysis Of The Fifth Bit (5th bit)

As previously pointed out, the resolution of the fifth bit is achieved on the basis of whether $(r_1{}^2-r_2{}^2)^4-24r_1{}^2r_2{}^2(r_1{}^2-r_2{}^2)+16r_1{}^4r_2{}^4$ exceeds or is exceeded by zero.

For example, from the decision logic for the fifth bit, as explained in Section 1, assuming $s_1(t)$ is transmitted,

$$\Pr\{\text{5th bit correct} / s_1(t)\} = \Pr\{(r_1^2 - r_2^2)^4 - 24r_1^2r_2^2(r_1^2 - r_2^2)^2 + 16r_1^4r_2^4 > 0 / s_1(t)\} \tag{3.95}$$

As has been indicated in the previous section, this form is not suitable for the noise analysis because of the non-linearity of the function involving r_1 and r_2. However, from Equation 3.5 it can be seen that

$$\Pr\{\text{5th bit correct} / s_1(t)\} = \Pr\{\cos 8\eta > 0\} \tag{3.96}$$

so that with the knowledge of the p.d.f of the angle η (which is already available), it is possible to evaluate this probability since it can be observed that $\cos 8\eta > 0$ for

$-\pi/16 \le \eta \le \pi/16$, $3\pi/16 \le \eta \le 5\pi/16$
$7\pi/16 \le \eta \le 9\pi/16$, $11\pi/16 \le \eta \le 13\pi/16$
$15\pi/16 \le \eta \le 17\pi/16$, $19\pi/16 \le \eta \le 21\pi/16$
$23\pi/16 \le \eta \le 25\pi/16$, $27\pi/16 \le \eta \le 29\pi/16$

By integrating the conditional p.d.f of η over these regions, Pr {5th bit correct / $s_1(t)$} can be appropriately obtained. Conditioned on $s_1(t)$ being transmitted, it has already been demonstrated that r_1 and r_2 are uncorrelated Gaussian random variables with

$$r_1 \sim N(\sqrt{E_s}\cos\alpha\, ,\, N_o/2)$$

and

$$r_2 \sim N(\sqrt{E_s}\sin\alpha\, ,\, N_o/2)$$

so that r_1 and r_2 are also statistically independent. It has also been indicated that

$$f_{\eta/s_1(t)}(H/s_1) = 1/2\pi \exp(-E_s/N_o)[1 + \sqrt{2\pi E_s/N_o/2}\, \exp(E_s/N_o \cos^2(H-\alpha)) \cos(H-\alpha)\, Q(-\sqrt{2E_s/N_o}\cos(H-\alpha))] \qquad 0 \le H \le 2\pi \tag{3.97}$$

so that

$$\Pr\{\text{5th bit correct}/s_1(t)\} = \Pr\{\cos 8\eta > 0/s_1(t)\} \tag{3.98}$$

$$\begin{aligned}
&= \int_{-\pi/16}^{\pi/16} f_{\eta/s_1(t)}(H/s_1)dH + \int_{3\pi/8}^{5\pi/16} f_{\eta/s_1(t)}(H/s_1)dH + \int_{7\pi/8}^{9\pi/16} f_{\eta/s_1(t)}(H/s_1)dH + \int_{-11\pi/8}^{13\pi/16} f_{\eta/s_1(t)}(H/s_1)dH \\
&+ \int_{15\pi/8}^{17\pi/16} f_{\eta/s_1(t)}(H/s_1)dH + \int_{19\pi/8}^{21\pi/16} f_{\eta/s_1(t)}(H/s_1)dH + \int_{23\pi/8}^{25\pi/16} f_{\eta/s_1(t)}(H/s_1)dH + \int_{27\pi/8}^{29\pi/16} f_{\eta/s_1(t)}(H/s_1)dH
\end{aligned}$$

The integrals can be modified by a change of variables to yield

$$\begin{aligned}
&= \int_{-\pi/16}^{\pi/16} [\, f_{\eta/s_1(t)}(H/s_1) + f_{\eta/s_1(t)}(H+\pi/2/s_1) + f_{\eta/s_1(t)}(H+\pi/s_1) \\
&\qquad + f_{\eta/s_1(t)}(H+3\pi/2/s_1)\,]\, dH \\
&+ \int_{3\pi/16}^{5\pi/16} [\, f_{\eta/s_1(t)}(H/s_1) + f_{\eta/s_1(t)}(H+\pi/2/s_1) + f_{\eta/s_1(t)}(H+\pi/s_1) \\
&\qquad + f_{\eta/s_1(t)}(H+3\pi/2/s_1)\,]\, dH
\end{aligned} \tag{3.99}$$

These expressions are identical to the ones dealt with in the analysis of the error probability associated with the 4th bit conditioned on $s_1(t)$ being transmitted. Using similar arguments to those used in relationship to the fourth bit noise performance analysis, we obtain

$$\begin{aligned}
&\Pr\{\text{5th bit correct}/s_1(t)\} = 1/2 \exp(-E_s/N_0) \\
&+ \sqrt{E_s/2\pi N_0/2}\, [\, \int_{-\pi/16}^{\pi/16} [\exp(-E_s/N_0 \sin^2(H-\alpha))\cos(H-\alpha) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha)) - Q(\sqrt{2E_s/N_0}\cos(H-\alpha))\} + \exp(-E_s/N_0 \cos^2(H-\alpha)) \\
&\sin(H-\alpha)\{Q(-\sqrt{2E_s/N_0}\sin(H-\alpha)) - Q(\sqrt{2E_s/N_0}\sin(H-\alpha))\}\,]dH \\
&+ \int_{3\pi/16}^{5\pi/16} [\exp(-E_s/N_0 \sin^2(H-\alpha))\cos(H-\alpha) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha)) - Q(\sqrt{2E_s/N_0}\cos(H-\alpha))\} + \exp(-E_s/N_0 \cos^2(H-\alpha)) \\
&\sin(H-\alpha)\{Q(-\sqrt{2E_s/N_0}\sin(H-\alpha)) - Q(\sqrt{2E_s/N_0}\sin(H-\alpha))\}]\, dH\,]
\end{aligned} \tag{3.100}$$

Because of the repetitive nature of the steps that must be carried out in order to obtain Pr {5th bit correct / $s_i(t)$}, i=2,3,----,8, only the pertinent results will be shown and all intermediate steps will be eliminated. Thus

$$\begin{aligned}
&\Pr\{\text{5th bit correct} / s_2(t)\} = \Pr\{\cos 8\eta < 0 / s_2(t)\} \\
&= 1 - P_r\{\cos 8\eta > 0 / s_2(t)\} = 1 - 1/2\exp(-E_s/N_0) \\
&-\sqrt{E_s/2\pi N_0/2}\,\Big[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\sin^2(H-\beta))\cos(H-\beta) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H-\beta))-Q(\sqrt{2E_s/N_0}\cos(H-\beta))\} + \exp(-E_s/N_0\cos^2(H-\beta)) \\
&\sin(H-\beta)\{Q(-\sqrt{2E_s/N_0}\sin(H-\beta))-Q(\sqrt{2E_s/N_0}\sin(H-\beta))\}\,]\,dH \\
&+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_0\sin^2(H-\beta))\cos(H-\beta) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H-\beta))-Q(\sqrt{2E_s/N_0}\cos(H-\beta))\} + \exp(-E_s/N_0\cos^2(H-\beta)) \\
&\sin(H-\beta)\{Q(-\sqrt{2E_s/N_0}\sin(H-\beta)) - Q(\sqrt{2E_s/N_0}\sin(H-\beta))\}]\,dH\,\Big]
\end{aligned} \tag{3.101}$$

Also

$$\begin{aligned}
&\Pr\{\text{5th bit correct} / s_3(t)\} = \Pr\{\cos 8\eta < 0 / s_3(t)\} \\
&= 1 - \Pr\{\cos 8\eta > 0 / s_3(t)\} = 1 - 1/2\exp(-E_s/N_0) \\
&-\sqrt{E_s/2\pi N_0/2}\,\Big[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\sin^2(H-\gamma))\cos(H-\gamma) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H-\gamma))-Q(\sqrt{2E_s/N_0}\cos(H-\gamma))\} + \exp(-E_s/N_0\cos^2(H-\gamma)) \\
&\sin(H-\gamma)\{Q(-\sqrt{2E_s/N_0}\sin(H-\gamma))-Q(\sqrt{2E_s/N_0}\sin(H-\gamma))\}\,]\,dH \\
&+\int_{3\pi/16}^{5\pi/16}\{\exp(-E_s/N_0\sin^2(H-\gamma))\cos(H-\gamma) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H-\gamma))-Q(\sqrt{2E_s/N_0}\cos(H-\gamma))\} + \exp(-E_s/N_0\cos^2(H-\gamma)) \\
&\sin(H-\gamma)\{Q(-\sqrt{2E_s/N_0}\sin(H-\gamma)) - Q(\sqrt{2E_s/N_0}\sin(H-\gamma))\}]\,dH\,\Big]
\end{aligned} \tag{3.102}$$

and

$$\Pr\{5\text{th bit correct}/s_4(t)\} = \Pr\{\cos 8\eta > 0/s_4(t)\} \tag{3.103}$$

$$= 1/2\exp(-E_s/N_0) + \sqrt{E_s/2\pi N_0/2}\,\Big[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\sin^2(H-\delta))\cos(H-\delta)$$
$$\{Q(-\sqrt{2E_s/N_0}\cos(H-\delta))-Q(\sqrt{2E_s/N_0}\cos(H-\delta))\} + \exp(-E_s/N_0\cos^2(H-\delta))$$
$$\sin(H-\delta)\{Q(-\sqrt{2E_s/N_0}\sin(H-\delta))-Q(\sqrt{2E_s/N_0}\sin(H-\delta))\}\,]\,dH$$
$$+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_0\sin^2(H-\delta))\cos(H-\delta)$$
$$\{Q(-\sqrt{2E_s/N_0}\cos(H-\delta))-Q(\sqrt{2E_s/N_0}\cos(H-\delta))\} + \exp(-E_s/N_0\cos^2(H-\delta))$$
$$\sin(H-\delta)\{Q(-\sqrt{2E_s/N_0}\sin(H-\delta)) - Q(\sqrt{2E_s/N_0}\sin(H-\delta))\}]\,dH\,\Big]$$

Similiarly,

$$\Pr\{5\text{th bit correct}/s_5(t)\} = \Pr\{\cos 8\eta > 0/s_5(t)\} \tag{3.104}$$

$$= 1/2\exp(-E_s/N_0) + \sqrt{E_s/2\pi N_0/2}\,\Big[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\cos^2(H+\delta))\sin(H+\delta)$$
$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\delta))-Q(\sqrt{2E_s/N_0}\sin(H+\delta))\} + \exp(-E_s/N_0\sin^2(H+\delta))$$
$$\cos(H+\delta)\{Q(-\sqrt{2E_s/N_0}\cos(H+\delta))-Q(\sqrt{2E_s/N_0}\cos(H+\delta))\}\,]\,dH$$
$$+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_0\cos^2(H+\delta))\sin(H+\delta)$$
$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\delta))-Q(\sqrt{2E_s/N_0}\sin(H+\delta))\} + \exp(-E_s/N_0\sin^2(H+\delta))$$
$$\cos(H+\delta)\{Q(-\sqrt{2E_s/N_0}\cos(H+\delta)) - Q(\sqrt{2E_s/N_0}\cos(H+\delta))\}]\,dH\,\Big]$$

Also,

$$\Pr\{5\text{th bit correct} / s_6(t)\} = \Pr\{\cos 8\eta < 0 / s_6(t)\}$$
$$= 1 - \Pr\{\cos 8\eta > 0/s_6(t)\} = 1 - 1/2\exp(-E_s/N_0)$$
$$-\sqrt{E_s/2\pi N_0/2}\,\Big[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\cos^2(H+\gamma))\sin(H+\gamma)$$
$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\gamma))-Q(\sqrt{2E_s/N_0}\sin(H+\gamma))\} + \exp(-E_s/N_0\sin^2(H+\gamma))$$
$$\cos(H+\gamma)\{Q(-\sqrt{2E_s/N_0}\cos(H+\gamma))-Q(\sqrt{2E_s/N_0}\cos(H+\gamma))\,]\,dH$$

$$\begin{aligned}&+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_0\mathrm{coa}^2(H+\gamma))\sin(H+\gamma)\\&\{Q(-\sqrt{2E_s/N_0}\sin(H+\gamma))-Q(\sqrt{2E_s/N_0}\sin(H+\gamma))\} + \exp(-E_s/N_0 \sin^2(H+\gamma))\\&\cos(H+\gamma)\{Q(-\sqrt{2E_s/N_0}\cos(H+\gamma)) - Q(\sqrt{2E_s/N_0}\cos(H+\gamma))\}]\, dH\,] \qquad (3.105)\end{aligned}$$

while

$$\begin{aligned}&\Pr\{\text{5th bit correct} / s_7(t)\} = \Pr\{\cos 8\eta < 0 / s_7(t)\}\\&= 1 - \Pr\{\cos 8\eta > 0 / s_7(t)\} = 1 - 1/2\exp(-E_s/N_0)\\&-\sqrt{E_s/2\pi N_0/2}\,[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\cos^2(H+\beta))\sin(H+\beta)\\&\{Q(-\sqrt{2E_s/N_0}\sin(H+\beta))-Q(\sqrt{2E_s/N_0}\sin(H+\beta))\} + \exp(-E_s/N_0 \sin^2(H+\beta))\\&\cos(H+\beta)\{Q(-\sqrt{2E_s/N_0}\cos(H+\beta))-Q(\sqrt{2E_s/N_0}\cos(H+\beta))\}]\, dH\\&+\int_{3\pi/16}^{5\pi/16}[e\ xp(-E_s/N_0\mathrm{coa}^2(H+\beta))\sin(H+\beta)\\&\{Q(-\sqrt{2E_s/N_0}\sin(H+\beta))-Q(\sqrt{2E_s/N_0}\sin(H+\beta))\} + \exp(-E_s/N_0 \sin^2(H+\beta))\\&\cos(H+\beta)\{Q(-\sqrt{2E_s/N_0}\cos(H+\beta)) - Q(\sqrt{2E_s/N_0}\cos(H+\beta))\}]\, dH\,] \qquad (3.106)\end{aligned}$$

and finally,

$$\begin{aligned}&\Pr\{\text{5th bit correct}/s_8(t)\} = \Pr\{\cos 8\eta > 0/s_8(t)\} \qquad (3.107)\\&= 1/2\exp(-E_s/N_0) + \sqrt{E_s/2\pi N_0/2}\,[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\cos^2(H+\alpha))\sin(H+\alpha)\\&\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha))\} + \exp(-E_s/N_0 \sin^2(H+\alpha))\\&\cos(H+\alpha)\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha))\}]\, dH\\&+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_0\cos^2(H+\alpha))\sin(H+\alpha)\\&\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha))\} + \exp(-E_s/N_0 \sin^2(H+\alpha))\\&\cos(H+\alpha)\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha)) - Q(\sqrt{2E_s/N_0}\cos(H+\alpha))\}]\, dH\,]\end{aligned}$$

As the equalities stated in Equation 3.22 are true for performance analysis involving the 5th bit also, a general expression for the probability of the fifth bit in error can be developed as previously done using these equalities and Equations 3.100 - 3.107 to yield

$$\Pr\{\text{5th bit correct}\} = \frac{1}{8}\sum_{i=1}^{8} \Pr\{\text{5th bit correct}/s_i(t)\} \tag{3.108}$$

so that the final form is

$$\begin{aligned}
\Pr\{\text{5th bit in error}\} &= 1-\Pr\{\text{5th bit correct}\} \\
&= 1/2 - \sqrt{E_s/\pi N_0}\,\Big[\int_0^{\pi/16} [\, \exp(-E_s/N_0 \sin^2(H-\alpha))
\end{aligned}$$

$$\begin{aligned}
&\cos(H-\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H-\alpha))\} \\
&+\exp(-E_s/N_0 \cos^2(H-\alpha)) \sin(H-\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H-\alpha))\} \\
&+\exp(-E_s/N_0 \sin^2(H+\alpha)) \cos(H+\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H+\alpha))\} \\
&+\exp(-E_s/N_0 \cos^2(H+\alpha)) \sin(H+\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H+\alpha)) \\
&-\exp(-E_s/N_0 \sin^2(H-\beta)) \cos(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H-\beta))\} \\
&-\exp(-E_s/N_0 \cos^2(H-\beta)) \sin(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H-\beta))\} \\
&-\exp(-E_s/N_0 \sin^2(H+\beta)) \cos(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H+\beta))\} \\
&-\exp(-E_s/N_0 \cos^2(H+\beta)) \sin(H+\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H+\beta))\} \\
&-\exp(-E_s/N_0 \sin^2(H-\gamma)) \cos(H-\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H-\gamma))\} \\
&-\exp(-E_s/N_0 \cos^2(H-\gamma)) \sin(H-\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H-\gamma))\} \\
&-\exp(-E_s/N_0 \sin^2(H+\gamma)) \cos(H+\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H+\gamma))\} \\
&-\exp(-E_s/N_0 \cos^2(H+\gamma)) \sin(H+\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H+\gamma))\} \\
&+\exp(-E_s/N_0 \sin^2(H-\delta)) \cos(H-\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H-\delta))\} \\
&+\exp(-E_s/N_0 \cos^2(H-\delta)) \sin(H-\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H-\delta))\} \\
&+\exp(-E_s/N_0 \sin^2(H+\delta)) \cos(H+\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H+\delta))\} + \\
&+\exp(-E_s/N_0 \cos^2(H+\delta)) \sin(H+\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H+\delta))\}\ ]\, dH \\
&+\int_{\pi/4}^{5\pi/16} [\, \exp(-E_s/N_0 \sin^2(H-\alpha))\cos(H-\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H-\alpha))\} \\
&+\exp(-E_s/N_0 \cos^2(H-\alpha)) \sin(H-\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H-\alpha))\} \\
&+\exp(-E_s/N_0 \sin^2(H+\alpha)) \cos(H+\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H+\alpha))\} \\
&+\exp(-E_s/N_0 \cos^2(H+\alpha)) \sin(H+\alpha)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H+\alpha)) \\
&-\exp(-E_s/N_0 \sin^2(H-\beta)) \cos(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H-\beta))\} \\
&-\exp(-E_s/N_0 \cos^2(H-\beta)) \sin(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\sin(H-\beta))\} \\
&-\exp(-E_s/N_0 \sin^2(H+\beta)) \cos(H-\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0}\cos(H+\beta))\}
\end{aligned}$$

$$
\begin{aligned}
&-\exp(-E_s/N_0 \cos^2(H+\beta)) \sin(H+\beta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H+\beta))\} \\
&-\exp(-E_s/N_0 \sin^2(H-\gamma)) \cos(H-\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H-\gamma))\} \\
&-\exp(-E_s/N_0 \cos^2(H-\gamma)) \sin(H-\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H-\gamma))\} \\
&-\exp(-E_s/N_0 \sin^2(H+\gamma)) \cos(H+\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H+\gamma))\} \\
&-\exp(-E_s/N_0 \cos^2(H+\gamma)) \sin(H+\gamma)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H+\gamma))\} \\
&+\exp(-E_s/N_0 \sin^2(H-\delta)) \cos(H-\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H-\delta))\} \\
&+\exp(-E_s/N_0 \cos^2(H-\delta)) \sin(H-\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H-\delta))\} \\
&+\exp(-E_s/N_0 \sin^2(H+\delta)) \cos(H+\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \cos(H+\delta))\} \\
&+\exp(-E_s/N_0 \cos^2(H+\delta)) \sin(H+\delta)\{1/4 - 1/2\, Q(\sqrt{2E_s/N_0} \sin(H+\delta))\}\ ]\, dH\ ] \\
&\qquad \equiv PE5 \qquad\qquad (3.110)
\end{aligned}
$$

6. Composite Bit Error Rate

The expressions for the probability of each bit in error have been developed and now the composite bit error rate can be obtained by adding all individual bit rates and then averaging them to yield the following result

$$\text{Bit Error Rate} = 1/5\,[PE1 + PE2 + PE3 + PE4 + PE5] \equiv PE \qquad (3.111)$$

where the appropriate terms making up this equation are given by Equations 3.28, 3.44, 3.68, 3.94 and 3.110.

This final expression has been evaluated on the computer for what are known to be the optimum phase parameter values, that is, $\alpha = \pi/32$, $\beta = 3\pi/32$, $\gamma = 5\pi/32$ and $\delta = 7\pi/32$.

The numerical results in the form of a graph are presented in Figure 3.3, showing the individual bit error rates and the composite bit error rate. A discussion of these numerical results and their significance is presented in Chapter 5.

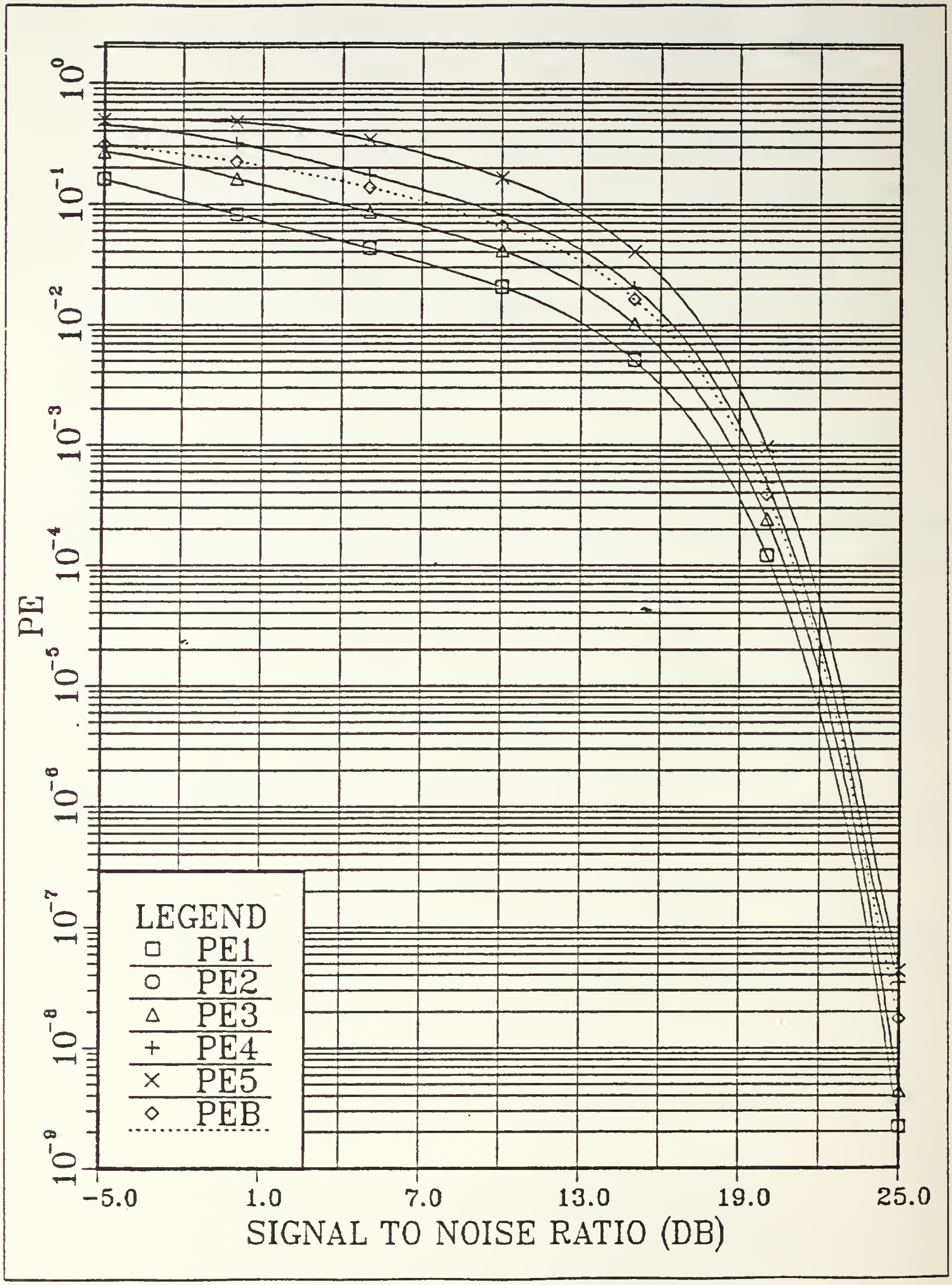

Figure 3.3 Receiver Noise Performance Curves for 32-PSK Modulated Signals.

IV. DIRECT BIT DETECTION RECEIVER FOR 64-PSK MODULATED SIGNALS

A. SIGNAL CONSTELLATION AND RECEIVER DESIGN

The concept of a direct bit detection receiver can now be extended to 64-PSK modulated signals. It has been stated previously that as M increases, the signal bandwidth decreases by a factor of 1/k relative to BPSK, where $k = \log_2 M$. Thus, the number of bits per symbol (and therefore per transmitted signal) is 6 for 64-PSK modulated signals. A signal constellation diagram with a Gray-Code bit to symbol mapping is shown in Figure 4.1.

A close examination of the signal space diagram of Figure 4.1 reveals that the first five bits of each symbol can be recovered using exactly the same logic as done in the 32-PSK case; hence, the same hardware can be used for such bit recovery. Extra hardware, however, will be required for the recovery of the sixth bit of each symbol, which as will be seen involves the use of only addition and multiplication of the terms already available in the implementation of the 32-PSK receiver.

It can be observed by careful examination of the signal space diagram of Figure 4.1 that for any reasonable choice of angles α_i, $i = 1,2,3,----,8$, the sign of the term $\cos 16\eta$ allows for the recovery of the sixth bit. That is, $\cos 16\eta > 0$ for the symbols in which the 6th bit is a '0' and $\cos 16\eta < 0$ for the symbols in which the 6th bit is a '1'. Clearly the factor a^{16} cannot affect the sign of this term, so decisions can be based on the sign of the term $a^{16} \cos 16\eta$, as this product is easily implementable by observing that

$$\begin{aligned} a^{16} \cos 16\eta &= a^{16} \cos(8\eta + 8\eta) \\ &= a^{16} \cos^2 8\eta - a^{16} \sin^2 8\eta \\ &= (a^8 \cos 8\eta)^2 - (a^8 \sin 8\eta)^2 \end{aligned} \tag{4.1}$$

The term '$a^8 \cos 8\eta$' is already available from the recovery of the fifth bit. Thus the term '$a^8 \sin 8\eta$' can be obtained by observing that

$$\begin{aligned} a^8 \sin 8\, \eta &= a^8\, 2 \sin 4\, \eta \cos 4\, \eta \\ &= 2\, \{4\, r_1\, r_2\, (r_1^2 - r_2^2)\}\{(r_1^2 - r_2^2) - 4\, r_1^2\, r_2^2\} \end{aligned} \tag{4.2}$$

so that

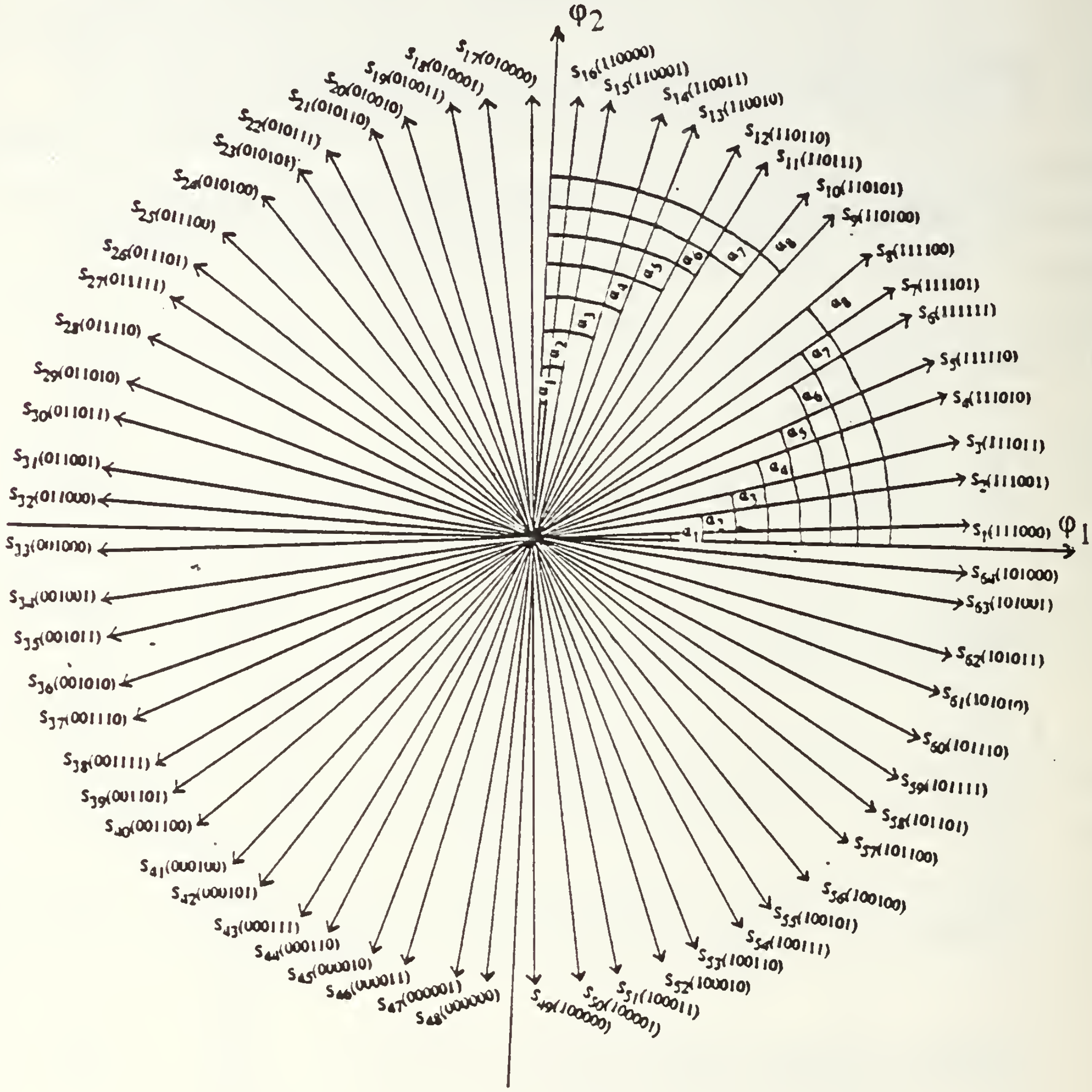

Figure 4.1 Signal Constellation for 64-PSK Modulated Signals with Gray Code Bit Assignment.

$$a^{16}\cos 16\eta = [(r_1^2 - r_2^2) - 24\, r_1^2 r_2^2 (r_1^2 - r_2^2) + 16\, r_1^4 r_2^4]^2$$
$$- [\, 2\, \{4\, r_1 r_2 (r_1^2 - r_2^2)\}\{(r_1^2 - r_2^2) - 4\, r_1^2 r_2^2\}\,]^2$$

All these terms are available from the recovery of the fifth and fourth bits and have already been implemented in the hardware of the receiver designed for 32-PSK modulated signals. Hence, taking advantage of this fact, the receiver for 64-PSK modulated signals can be implemented in hardware by using the terms generated directly from the implementation of the receiver for 32-PSK modulated signals. The complete design of the 64-PSK receiver is shown in Figure 4.2.

B. NOISE PERFORMANCE ANALYSIS

The interference model to be used in this receiver performance analysis is the previously described Additive White Gaussian Noise model.

Using the symmetry properties of the signal vectors and the fact that all signals are assumed equiprobable, the bit error rate for each bit can be obtained by considering only the cases in which the signals in the first quadrant, that is, $s_i(t)$, $i = 1,2,3,\text{---},16$, are assumed transmitted. The determination of the BER for five out of the six bits that make up each symbol is identical to the method used in the previous chapter, except for slight notational changes. Therefore, most derivation steps will be left out and only intermediate and final results will be presented.

The transmitted signals, which are represented as vectors in Figure 4.1, can be mathematically expressed as

$$s_i(t) = \sqrt{2E_s/T_s}\,\sin[2\pi f_c t + \theta_i(t)] \quad 0 \le t \le T_s \quad i = 1,2,3,\text{----},64 \tag{4.3}$$

or

$$s_i(t) = \sqrt{E_s}\cos\theta_i(t)\varphi_1(t) + \sqrt{E_s}\sin\theta_i(t)\varphi_2(t) \qquad 0 \le t \le T_s$$
$$i = 1,2,\text{------},64 \tag{4.4}$$

where $\theta_i(t)$ equals

$\alpha_1 + (i-1)\pi/32$ for $i = 1,17,33,49$
$\alpha_2 + (i-2)\pi/32$ for $i = 2,18,34,50$
$\alpha_3 + (i-3)\pi/32$ for $i = 3,19,35,51$
$\alpha_4 + (i-4)\pi/32$ for $i = 4,20,36,52$
$\alpha_5 + (i-5)\pi/32$ for $i = 5,21,37,53$
$\alpha_6 + (i-6)\pi/32$ for $i = 6,22,38,54$
$\alpha_7 + (i-7)\pi/32$ for $i = 7,23,39,55$

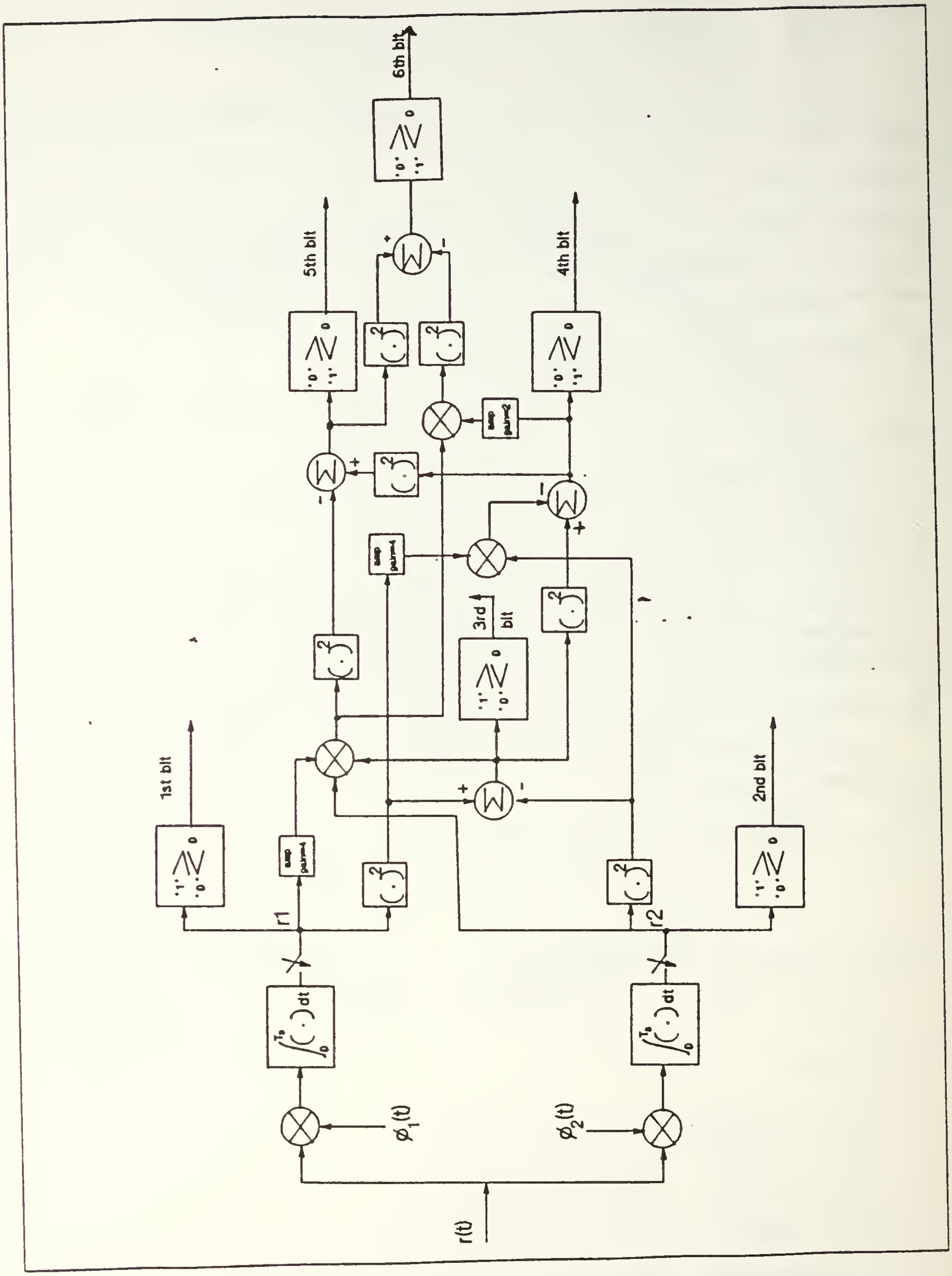

Figure 4.2 Proposed Receiver Structure for 64-PSK Modulated Signals.

$\alpha_8 + (i-8)\pi/32$ for $i = 8,24,40,56$
$(i+7)\pi/32 - \alpha_8$ for $i = 9,25,41,57$
$(i+6)\pi/32 - \alpha_7$ for $i = 10,26,42,58$
$(i+5)\pi/32 - \alpha_6$ for $i = 11,27,43,59$
$(i+4)\pi/32 - \alpha_5$ for $i = 12,28,44,60$
$(i+3)\pi/32 - \alpha_4$ for $i = 13,29,45,61$
$(i+2)\pi/32 - \alpha_3$ for $i = 14,30,46,62$
$(i+1)\pi/32 - \alpha_2$ for $i = 15,31,47,63$
$(i)\pi/32 - \alpha_1$ for $i = 16,32,48,64$

It is assumed that every T_s seconds we receive the signal r(t) which consists of the transmitted signal $s_i(t)$ and the interference n(t), namely

$$r(t) = s_i(t) + n(t) \tag{4.5}$$

where $s_i(t)$ can be expressed in terms of basis functions $\varphi_1(t)$ and $\varphi_2(t)$ as given by Equation 4.4 and n(t) is a sample function of the Additive White Gaussian Noise (AWGN) having power spectral density level $N_0/2$. Using the concepts and procedures utilized in the previous chapter, we can now carry out the analysis of the BER of each bit separately, and then combine all the results in the end.

1. Noise Analysis Of The First Bit (MSB)

If we assume that $s_i(t)$, $i=1,2,3,\text{----},8$ is transmitted, then from the signal space diagram it can be observed that

$$\Pr\{\text{MSB correct} / s_i(t)\} = \Pr\{ r_1 > 0 / s_i(t) \} \quad i=1,2,3,\text{---},8 \tag{4.6}$$

From the work done in the previous chapter (see Equations 3.10 -3.13) we obtain

$$\Pr\{ r_1 > 0 / s_i(t) \} = \int_{-\sqrt{2E_s/N_0}\cos\alpha_i}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-X^2/2)\, dX \tag{4.7}$$

$$= Q(-\sqrt{2E_s/N_0}\cos\alpha_i) \quad i = 1,2,3,\text{----},8$$

Similarly

$$\Pr\{\text{MSB correct} / s_i(t)\} = Q(-\sqrt{2E_s/N_0}\cos(\pi/2 - \alpha_{17-i}))$$
$$i = 9,10,\text{------},16$$

which can be equivalently written as

$$\Pr\{\text{MSB correct} / s_i(t)\} = Q(-\sqrt{2E_s/N_o}\,\sin\alpha_{17-i}) \quad i = 9,10,\text{----},16 \qquad (4.8)$$

As explained in the previous chapter, due to rotational symmetry of the signals it can be stated that

$$\Pr\{\text{MSB correct} / s_k(t)\} = \Pr\{\text{MSB correct} / s_i(t)\} \qquad (4.9)$$

where for

$k = 1$, $i = 32,33,64$
$k = 2$, $i = 31,34,63$
$k = 3$, $i = 30,35,62$
$k = 4$, $i = 29,36,61$
$k = 5$, $i = 28,37,60$
$k = 6$, $i = 27,38,59$
$k = 7$, $i = 26,39,58$
$k = 8$, $i = 25,40,57$
$k = 9$, $i = 24,41,56$
$k = 10$, $i = 23,42,55$
$k = 11$, $i = 22,43,54$
$k = 12$, $i = 21,44,53$
$k = 13$, $i = 20,45,52$
$k = 14$, $i = 19,46,51$
$k = 15$, $i = 18,47,49$
$k = 16$, $i = 17,48,48$

Due to the equalities given by Equation 4.9 it is clear that

$$\Pr\{\text{MSB correct}\} = \frac{1}{16}\sum_{i=1}^{16} \Pr\{\text{MSB correct}/s_i(t)\} \qquad (4.10)$$

Using Equations 4.6 and 4.8 and simplifying whenever possible results in the final expression

$$\Pr\{\text{MSB in error}\} = 1/16\,\Big[\sum_{i=1}^{8} Q(\sqrt{2E_s/N_o}\cos\alpha_i) + \sum_{i=1}^{8} Q(\sqrt{2E_s/N_o}\sin\alpha_i)\Big] \equiv \text{PE1} \qquad (4.11)$$

2. Noise Analysis Of The Second Bit (2nd bit)

If we assume that the signal $s_i(t)$, $i = 1,2,3,----,8$ is transmitted, from the signal space diagram, it can be observed that

$$\Pr\{\text{2nd bit correct} / s_i(t)\} = \Pr\{ r_2 > 0 / s_i(t) \} \tag{4.12}$$
$$= Q(-\sqrt{2E_s/N_0}\ \sin\alpha_i) \quad i = 1,2,3,----,8$$

and similarly

$$\Pr\{\text{2nd bit correct} / s_i(t) \} = Q(-\sqrt{2E_s/N_0}\ \sin(\pi/2 - \alpha_{17-i}))$$
$$i = 9,10,------,16$$

which can be equivalently expressed as

$$\Pr\{\text{2nd bit correct} / s_i(t)\} = Q(-\sqrt{2E_s/N_0}\ \cos\alpha_{17-i}) \quad i = 9,10,----,16 \tag{4.13}$$

The symmetry of the signals which leads to equalities of probability of a bit being correctly recovered, and expressed in Equation 4.9, for the first bit, holds for the 2nd bit also, so that using these equalities we can write

$$\Pr\{\text{2nd bit correct}\} = \frac{1}{16} \sum_{i=1}^{16} \Pr\{\text{2nd bit correct}/s_i(t)\} \tag{4.14}$$

Using Equations 4.12 and 4.13 and simplifying whenever possible results in the final expression

$$\Pr\{\text{2nd bit in error}\} = 1/16 \left[\sum_{i=1}^{8} Q(\sqrt{2E_s/N_0}\sin\alpha_i) + \sum_{i=1}^{8} Q(\sqrt{2E_s/N_0}\cos\alpha_i)\right] \equiv PE2 \tag{4.15}$$

Examination of the resulting expressions for PE1 and PE2, shows that PE1 = PE2 as expected

3. Noise Analysis Of The Third Bit (3rd bit)

From the signal space diagram of Figure 4.1, it can be observed that for the third bit to be correctly recovered, assuming that $s_i(t)$, $i = 1,2,3,-----,8$ is transmitted,

$$(r_1^2 - r_2^2) > 0 \tag{4.16}$$

must hold, while

$$(r_1^2 - r_2^2) < 0 \tag{4.17}$$

must be satisfied if $s_i(t)$, $i = 9,10,\text{----},16$ is transmitted

Assume that $s_i(t)$, $i = 1,2,\text{------},8$ is the trasmitted signal, so that

$$\begin{aligned} \Pr\{\text{3rd bit correct} / s_i(t)\} &= \Pr\{(r_1^2 - r_2^2) > 0 / s_i(t)\} \\ &= \Pr\{(r_1 + r_2)(r_1 - r_2) > 0 / s_i(t)\} \\ & i = 1,2,\text{----},8 \end{aligned} \tag{4.18}$$

Define under the condition of the signal $s_i(t)$ transmitted $i = 1,2,\text{----},8$,

$$X = r_1 + r_2 \quad \text{and} \quad Y = r_1 - r_2 \tag{4.19}$$

It has been demonstrated in the previous chapter that X and Y are conditionally independent Gaussian random variables. Hence we can write

$$\begin{aligned} \Pr\{(r_1 + r_2)(r_1 - r_2) > 0 / s_i(t)\} &= \Pr\{XY > 0 / s_i(t)\} \\ &= \Pr\{X > 0 / s_i(t)\} \Pr\{Y > 0 / s_i(t)\} + \\ &\quad + \Pr\{X < 0 / s_i(t)\} \Pr\{Y < 0 / s_i(t)\} \\ & i = 1,2,\text{-----},8 \end{aligned} \tag{4.20}$$

This type of expression has already been worked out before so that using available results we obtain

$$\begin{aligned} \Pr\{\text{3rd bit correct} / s_i(t)\} &= Q(-\sqrt{E_s/N_0}(\cos\alpha_i + \sin\alpha_i))Q(-\sqrt{E_s/N_0}(\cos\alpha_i - \sin\alpha_i)) \\ &+ \{1-Q(-\sqrt{E_s/N_0}(\cos\alpha_i + \sin\alpha_i))\}\{1-Q(-\sqrt{E_s/N_0}(\cos\alpha_i - \sin\alpha_i))\} \\ & i = 1,2,\text{-----},8 \end{aligned} \tag{4.21}$$

and similarly, when conditioning on $s_{i(t)}$, $i = 9,10\ \text{-----},16$ being transmitted

$$\begin{aligned} \Pr\{\text{3rd bit correct} / s_i(t)\} &= \Pr\{(r_1^2 - r_2^2) < 0 / s_i(t)\} \\ &= 1 - \Pr\{(r_1^2 - r_2^2) > 0 / s_i(t)\} \\ &= 1 - Q(-\sqrt{E_s/N_0}(\sin\alpha_{17-i} + \cos\alpha_{17-i}))Q(-\sqrt{E_s/N_0}(\sin\alpha_{17-i} - \cos\alpha_{17-i})) \\ &\quad -\{1-Q(-\sqrt{E_s/N_0}(\sin\alpha_{17-i} + \cos\alpha_{17-i}))\}\{1-Q(-\sqrt{E_s/N_0}(\sin\alpha_{17-i} - \cos\alpha_{17-i}))\} \\ & i = 9,10,\text{-------},16 \end{aligned} \tag{4.22}$$

The equalities given by Equation 4.9 hold for $\Pr\{\text{3rd bit correct} / s_i(t)\}$, so that we can obtain

$$\Pr\{\text{3rd bit correct}\} = \frac{1}{16} \sum_{i=1}^{16} \Pr\{\text{3rd bit correct}/s_i(t)\} \tag{4.23}$$

so that using Equations 4.21 and 4.22 and simplifying whenever possible, results in

$$\Pr\{\text{3rd bit in error}\} = 1/8\,[\sum_{i=1}^{8} Q(\sqrt{E_s/N_o}(\cos\alpha_i - \sin\alpha_i)) + Q(\sqrt{E_s/N_o}(\cos\alpha_i + \sin\alpha_i))$$
$$\{1/8 - 1/4\, Q(\sqrt{E_s/N_o}(\cos\alpha_i - \sin\alpha_i))\}\,] \equiv PE3 \qquad (4.24)$$

4. Noise Analysis Of The Fourth Bit (4th bit)

The decision rule for the resolution of the fourth bit was explained in the Chapter III, Section B-4. Therefore, the probability that this bit is correctly recovered given that $s_i(t)$, $i = 1,2,3,4,13,14,15,16$ was transmitted is given by

$$\Pr\{\text{4th bit correct} / s_i(t)\} = \Pr\{(r_1^2 - r_2^2)^2 - 4r_1^2 r_2^2 > 0 / s_i(t)\}$$
$$i = 1,2,3,4,13,14,15,16 \qquad (4.25)$$

or equivalently (as also previously explained)

$$\Pr\{\text{4th bit correct} / s_i(t)\} = \Pr\{a^4 \cos 4\eta > 0 / s_i(t)\} \qquad (4.26)$$
$$= \Pr\{\cos 4\eta > 0 / s_i(t)\}$$

This probability can be evaluated using the p.d.f of the angle 'η' conditioned on $s_i(t)$, $i = 1,2,3,4,13,14,15,16$ being transmitted, namely

$$f_{\eta/s_i(t)}(H/s_i) = 1/2\pi \exp(-E_s/N_o)[1 + \sqrt{2\pi E_s/N_o}/2 \exp(E_s/N_o \cos^2(H-\alpha_i))$$
$$\cos(H-\alpha_i)\, Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))] \qquad 0 \le H \le 2\pi \qquad (4.27)$$
$$i = 1,2,3,4$$

and since

$$\cos 4\eta > 0\,, \qquad \text{for } -\pi/8 \le \eta \le \pi/8\,,\ 3\pi/8 \le \eta \le 5\pi/8$$
$$\text{and } \quad 7\pi/8 \le \eta \le 9\pi/8\,,\ 11\pi/8 \le \eta \le 13\pi/8$$

we obtain

$$\Pr\{\text{4th bit correct}/s_i(t)\} = \Pr\{\cos 4\eta > 0/s_i(t)\} \qquad (4.28)$$
$$= \int_{-\pi/8}^{\pi/8} f_{\eta/s_i(t)}(H/s_i)dH + \int_{3\pi/8}^{5\pi/8} f_{\eta/s_i(t)}(H/s_i)dH + \int_{7\pi/8}^{9\pi/8} f_{\eta/s_i(t)}(H/s_i)dH + \int_{11\pi/8}^{13\pi/8} f_{\eta/s_i(t)}(H/s_i)dH$$

$$= \int_{-\pi/8}^{\pi/8} [f_{\eta/s_i(t)}(H/s_i) + f_{\eta/s_i(t)}(H+\pi/2/s_i) + f_{\eta/s_i(t)}(H+\pi/s_i) + f_{\eta/s_i(t)}(H+3\pi/2/s_i)\,]dH$$
$$i = 1,2,3,4 \qquad (4.29)$$

The individual terms of Equation 4.29 can be obtained by using Equation 4.27, namely

$$\begin{aligned}&f_{\eta/s_i(t)}\,(H+\pi/2/s_i) = 1/2\pi \exp(-E_s/N_0)[1-\sqrt{2\pi E_s/N_0}/2\,\exp(E_s/N_0 \sin^2(H-\alpha_i))\\ &\sin(H-\alpha_i)\;Q(-\sqrt{2E_s/N_0}\sin(H-\alpha_i))] \quad i = 1,2,3,4 \qquad (4.30)\end{aligned}$$

$$\begin{aligned}&f_{\eta/s_i(t)}\,(H+\pi/s_i) = 1/2\pi \exp(-E_s/N_0)[1-\sqrt{2\pi E_s/N_0}/2\,\exp(E_s/N_0 \cos^2(H-\alpha_i))\\ &\cos(H-\alpha_i)\;Q(-\sqrt{2E_s/N_0}\cos(H-\alpha_i))] \quad i = 1,2,3,4 \qquad (4.31)\end{aligned}$$

$$\begin{aligned}&f_{\eta/s_i(t)}\,(H+3\pi/2/s_i) = 1/2\pi \exp(-E_s/N_0)[1+\sqrt{2\pi E_s/N_0}/2\,\exp(E_s/N_0 \sin^2(H-\alpha_i))\\ &\sin(H-\alpha_i)\;Q(-\sqrt{2E_s/N_0}\sin(H-\alpha_i))] \quad i = 1,2,3,4 \qquad (4.32)\end{aligned}$$

Substituting these expressions in Equation 4.29 and simplifying by adding the appropriate terms, we obtain

$$\begin{aligned}&\Pr\{\text{4th bit correct}/s_i(t)\} = 1/2\exp(-E_s/N_0)\\ &+ \sqrt{E_s/2\pi N_0}/2 \int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_0 \sin^2(H-\alpha_i))\cos(H-\alpha_i)\\ &\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\cos(H-\alpha_i))\}\\ &+\exp(-E_s/N_0\,\cos^2(H-\alpha_i))\sin(H-\alpha_i)\\ &\{Q(-\sqrt{2E_s/N_0}\,\sin(H-\alpha_i)) - Q(\sqrt{2E_s/N_0}\,\sin(H-\alpha_i))\}]\,dH\\ &\qquad i = 1,2,3,4 \qquad (4.33)\end{aligned}$$

and

$$\begin{aligned}&\Pr\{\text{4th bit correct}/s_i(t)\} = 1/2\exp(-E_s/N_0)\\ &+\sqrt{E_s/2\pi N_0/2} \int_{-\pi/8}^{\pi/8}[\exp(-E_s/N_0 \cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})\\ &\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}\\ &+\exp(-E_s/N_0 \sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})\\ &\{Q(-\sqrt{2E_s/N_0}\,\cos(H+\alpha_{17-i})) - Q(\sqrt{2E_s/N_0}\,\cos(H+\alpha_{17-i}))\}]\,dH\\ &\qquad i = 13,14,15,16 \qquad (4.34)\end{aligned}$$

Assume now that $s_i(t)$, $i = 5,6,7,8,9,10,11,12$ is transmitted, so that from Figure 4.1

$$\begin{aligned}\Pr\{\text{4th bit correct} / s_i(t)\} &= \Pr\{\cos 4\eta < 0 / s_i(t)\} \\ &= 1 - \Pr\{\cos 4\eta > 0 / s_i(t)\} \\ i &= 5,6,7,8,9,10,11,12\end{aligned} \tag{4.35}$$

From the expressions already developed, we can write

$$\begin{aligned}&\Pr\{\text{4th bit correct}/s_i(t)\} = 1 - 1/2 \exp(-E_s/N_0) \\ &-\sqrt{E_s/2\pi N_0}/2 \int_{-\pi/8}^{\pi/8} [\exp(-E_s/N_0 \sin^2(H-\alpha_i))\cos(H-\alpha_i) \\ &\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\cos(H-\alpha_i))\} \\ &+\exp(-E_s/N_0 \cos^2(H-\alpha_i))\sin(H-\alpha_i) \\ &\{Q(-\sqrt{2E_s/N_0} \sin(H-\alpha_i)) - Q(\sqrt{2E_s/N_0} \sin(H-\alpha_i))\}]\, dH \\ &\qquad i = 5,6,7,8\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}&\Pr\{\text{4th bit correct}/s_i(t)\} = 1 - 1/2 \exp(-E_s/N_0) \\ &-\sqrt{E_s/2\pi N_0}/2 \int_{-\pi/8}^{\pi/8} [\exp(-E_s/N_0 \cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i}) \\ &\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\} \\ &+\exp(-E_s/N_0 \sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i}) \\ &\{Q(-\sqrt{2E_s/N_0} \cos(H+\alpha_{17-i})) - Q(\sqrt{2E_s/N_0} \cos(H+\alpha_{17-i}))\}]\, dH \\ &\qquad i = 9,10,11,12\end{aligned} \tag{4.37}$$

Now using the equalities similar to those in Equation 4.9 we obtain

$$\Pr\{\text{4th bit correct}\} = \frac{1}{16} \sum_{i=1}^{16} \Pr\{\text{4th bit correct}/s_i(t)\} \tag{4.38}$$

so that using Equations 4.33, 4.34, 4.36 and 4.37 and simplifying by collecting the appropriate terms we obtain

$$\Pr\{\text{4th bit in error}\} = 1/2 - \sqrt{E_s/\pi N_o} \int_0^{\pi/8} [\sum_{i=1}^{8} \rho_i\{\exp(-E_s/N_o \sin^2(H-\alpha_i))$$

$$\cos(H-\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\}$$

$$+\exp(-E_s/N_o \cos^2(H-\alpha_i)) \sin(H-\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))\}$$

$$+\exp(-E_s/N_o \sin^2(H+\alpha_i)) \cos(H+\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_o}\cos(H+\alpha_i))\}$$

$$+\exp(-E_s/N_o \cos^2(H+\alpha_i)) \sin(H+\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_o}\sin(H+\alpha_i))\}]\, dH$$

$$\equiv PE4 \qquad (4.39)$$

where ρ_i is defined as

$$\rho_i = \begin{cases} 1 & \text{when } i = 1,2,3,4 \\ -1 & \text{when } i = 5,6,7,8 \end{cases}$$

5. Noise Analysis Of The Fifth Bit (5th bit)

The decision rule for the recovery of the fifth bit was explained in Chapter III, Section B-5. and therefore the probability that this bit is correctly recovered, given that $s_i(t)$, $i = 1,2,7,8,9,10,15,16$ was transmitted, is given by

$$\Pr\{\text{5th bit correct} / s_i(t)\} = \Pr\{a^8 \cos 8\eta > 0 / s_i(t)\} \qquad (4.40)$$

$$= \Pr\{\cos 8\eta > 0 / s_i(t)\}$$

As shown in previous work, this probability can be evaluated using the p.d.f of the angle 'η' conditioned on $s_i(t)$, $i = 1,2,7,8$ being transmitted, which as explained before, is given by

$$f_{\eta/s_i(t)}(H/s_i) = 1/2\pi \exp(-E_s/N_o)[1 + \sqrt{2\pi E_s/N_o}/2 \exp(E_s/N_o \cos^2(H-\alpha_i))$$

$$\cos(H-\alpha_i)\, Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))] \qquad 0 \le H \le 2\pi \qquad i = 1,2,7,8 \qquad (4.41)$$

and since $\cos 8\eta > 0$ for

$$-\pi/16 \le \eta \le \pi/16\,,\ 3\pi/16 \le \eta \le 5\pi/16$$

$$7\pi/16 \le \eta \le 9\pi/16\,,\ 11\pi/16 \le \eta \le 13\pi/16$$

$$15\pi/16 \le \eta \le 17\pi/16\,,\ 19\pi/16 \le \eta \le 21\pi/16$$

$$23\pi/16 \le \eta \le 25\pi/16\,,\ 27\pi/16 \le \eta \le 29\pi/16$$

it is clear that

$$\Pr\{5\text{th bit correct}/s_i(t)\} = \Pr\{\cos 8\eta > 0/s_i(t)\} \qquad (4.42)$$

$$= \int_{-\pi/16}^{\pi/16} f_{\eta/s_i(t)}(H/s_i)dH + \int_{3\pi/8}^{5\pi/16} f_{\eta/s_i(t)}(H/s_i)dH + \int_{7\pi/8}^{9\pi/16} f_{\eta/s_i(t)}(H/s_i)dH + \int_{11\pi/8}^{13\pi/16} f_{\eta/s_i(t)}(H/s_i)dH$$

$$+ \int_{15\pi/8}^{17\pi/16} f_{\eta/s_i(t)}(H/s_i)dH + \int_{19\pi/8}^{21\pi/16} f_{\eta/s_i(t)}(H/s_i)dH + \int_{23\pi/8}^{25\pi/16} f_{\eta/s_i(t)}(H/s_i)dH + \int_{27\pi/8}^{29\pi/16} f_{\eta/s_i(t)}(H/s_i)dH$$

$$i = 1,2,7,8$$

This can be simplified via variable changes, namely

$$= \int_{-\pi/16}^{\pi/16} [f_{\eta/s_i(t)}(H/s_i) + f_{\eta/s_i(t)}(H+\pi/2/s_i) + f_{\eta/s_i(t)}(H+\pi/s_i) + f_{\eta/s_i}(t)(H+3\pi/2/s_i)]dH$$

$$+ \int_{3\pi/16}^{5\pi/16} [f_{\eta/s_i(t)}(H/s_i) + f_{\eta/s_i(t)}(H+\pi/2/s_i) + f_{\eta/s_i(t)}(H+\pi/s_i) + f_{\eta/s_i(t)}(H+3\pi/2/s_i)]dH$$

$$i = 1,2,7,8 \qquad (4.43)$$

These expressions are identical to the ones dealt with in the performance analysis involving the 4th bit, so that

$$\Pr\{5\text{th bit correct}/s_i(t)\} = 1/2\exp(-E_s/N_o)$$

$$+ \sqrt{E_s/2\pi N_o}/2\ [\int_{-\pi/16}^{\pi/16} [\exp(-E_s/N_o\sin^2(H-\alpha_i))\cos(H-\alpha_i)$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i)) - Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\} + \exp(-E_s/N_o\cos^2(H-\alpha_i))$$

$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i)) - Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))]\ dH$$

$$+ \int_{3\pi/16}^{5\pi/16} [\exp(-E_s/N_o\sin^2(H-\alpha_i))\cos(H-\alpha_i)$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i)) - Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\} + \exp(-E_s/N_o\cos^2(H-\alpha_i))$$

$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i)) - Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))\}]\ dH\ ]$$

$$i = 1,2,7,8 \qquad (4.44)$$

and similarly

$$\Pr\{\text{5th bit correct}/s_i(t)\} = 1/2\exp(-E_s/N_o)$$

$$+\sqrt{E_s/2\pi N_o/2}\;[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_o\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_o}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_o}\sin(H+\alpha_{17-i}))\}$$

$$+\exp(-E_s/N_o\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_o}\cos(H+\alpha_{17-i}))]\,dH$$

$$+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_o\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_o}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_o}\sin(H+\alpha_{17-i}))\}$$

$$+\exp(-E_s/N_o\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_o}\cos(H+\alpha_{17-i}))]\,dH$$

$$i = 9,10,15,16 \qquad (4.45)$$

Assume now that $s_i(t)$, $i = 3,4,5,6,11,12,13,14$ is transmitted, so that from the explaination given on page 30.

$$\Pr\{\text{5th bit correct}/s_i(t)\} = \Pr\{\cos 8\eta < 0/s_i(t)\} \qquad (4.46)$$

$$= 1-\Pr\{\cos 8\eta > 0/s_i(t)\}$$

$$i = 3,4,5,6,11,12,13,14$$

From the expressions already developed we can write

$$\Pr\{\text{5th bit correct}/s_i(t)\} = 1-1/2\exp(-E_s/N_o)$$

$$-\sqrt{E_s/2\pi N_o/2}\;[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_o\sin^2(H-\alpha_i))\cos(H-\alpha_i)$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\}+\exp(-E_s/N_o\cos^2(H-\alpha_i))$$

$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))]\,dH$$

$$+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_o\sin^2(H-\alpha_i))\cos(H-\alpha_i)$$

$$\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\} + \exp(-E_s/N_o\cos^2(H-\alpha_i))$$

$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i)) - Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))\}]\,dH\,]$$

$$i = 3,4,5,6 \qquad (4.47)$$

and similarly

$$
\begin{aligned}
&\Pr\{\text{5th bit correct}/s_i(t)\} = 1 - 1/2\exp(-E_s/N_0) \\
&-\sqrt{E_s/2\pi N_0/2}\,\Big[\int_{-\pi/16}^{\pi/16}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i}) \\
&\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\} \\
&+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i}) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]\,dH \\
&+\int_{3\pi/16}^{5\pi/16}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i}) \\
&\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\} \\
&+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i}) \\
&\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]\,dH \\
&\quad i = 11,12,13,14 \qquad (4.48)
\end{aligned}
$$

Now using the equalities in Equation 4.9 as they are applicable to the present analysis, we obtain

$$\Pr\{\text{5th bit correct}\} = \frac{1}{16}\sum_{i=1}^{16}\Pr\{\text{5th bit correct}/s_i(t)\} \qquad (4.49)$$

so that using Equations 4.44, 4.45, 4.47 and 4.48 and simplifying by collecting the appropriate terms we obtain

$$\begin{aligned}
\Pr\{5\text{th bit in error}\} = 1/2 - \sqrt{E_s/\pi N_0}\,[\int_{0}^{\pi/16} [\sum_{i=1}^{8} \mu_i\{&\exp(-E_s/N_0 \sin^2(H-\alpha_i)) \\
&\cos(H-\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \cos(H-\alpha_i))\} \\
&+\exp(-E_s/N_0 \cos^2(H-\alpha_i)) \sin(H-\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \sin(H-\alpha_i))\} \\
&+\exp(-E_s/N_0 \sin^2(H+\alpha_i)) \cos(H+\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \cos(H+\alpha_i))\} \\
&+\exp(-E_s/N_0 \cos^2(H+\alpha_i)) \sin(H+\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \sin(H+\alpha_i))\}] \\
&+\int_{\pi/4}^{5\pi/16} [\sum_{i=1}^{8} \mu_i\{\exp(-E_s/N_0 \sin^2(H-\alpha_i)) \\
&\cos(H-\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \cos(H-\alpha_i))\} \\
&+\exp(-E_s/N_0 \cos^2(H-\alpha_i)) \sin(H-\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \sin(H-\alpha_i))\} \\
&+\exp(-E_s/N_0 \sin^2(H+\alpha_i)) \cos(H+\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \cos(H+\alpha_i))\} \\
&+\exp(-E_s/N_0 \cos^2(H+\alpha_i)) \sin(H+\alpha_i)\{1/8 - 1/4\, Q(\sqrt{2E_s/N_0} \sin(H+\alpha_i))\}]\,] \\
&\equiv PE5 \qquad (4.50)
\end{aligned}$$

where μ_i is defined follows

$$\mu_i = \begin{cases} 1 & \text{when } i = 1,2,7,8 \\ -1 & \text{when } i = 3,4,5,6 \end{cases}$$

6. Noise Analysis Of The Sixth Bit (6th bit)

From the decision rule for the recovery of the sixth bit, as explained in Section 1 of this Chapter, the probability that this bit is correctly recovered given that $s_i(t)$, $i = 1,4,5,8,9,12,13,16$ was transmitted is given by

$$\begin{aligned}
\Pr\{6\text{th bit correct} / s_i(t)\} &= \Pr\{a^{16} \cos 16\eta > 0 / s_i(t)\} \qquad (4.51) \\
&= \Pr\{\cos 16\eta > 0 / s_i(t)\}
\end{aligned}$$

This probability, as explained before, can be evaluated by the knowledge of the p.d.f of the angle 'η' conditioned on $s_i(t)$, $i = 1,4,5,8$ being transmitted, namely

$$\begin{aligned}
f_{\eta/s_i(t)}(H/s_i) = {} & 1/2\pi \exp(-E_s/N_0)[1 + \sqrt{2\pi E_s/N_0/2}\, \exp(E_s/N_0 \cos^2(H-\alpha_i)) \\
& \cos(H-\alpha_i)\, Q(-\sqrt{2E_s/N_0} \cos(H-\alpha_i))] \qquad 0 \le H \le 2\pi \qquad (4.52) \\
& i = 1,4,5,8
\end{aligned}$$

Since, $\cos 16\eta \ge 0$ for

$-\pi/32 \le \eta \le \pi/32$, $3\pi/32 \le \eta \le 5\pi/32$

$7\pi/32 \le \eta \le 9\pi/32$, $11\pi/32 \le \eta \le 13\pi/32$

$$15\pi/32 \le \eta \le 17\pi/32\,,\; 19\pi/32 \le \eta \le 21\pi/32$$
$$23\pi/32 \le \eta \le 25\pi/32\,,\; 27\pi/32 \le \eta \le 29\pi/32$$
$$31\pi/32 \le \eta \le 33\pi/32\,,\; 35\pi/32 \le \eta \le 37\pi/32$$
$$39\pi/32 \le \eta \le 41\pi/32\,,\; 43\pi/32 \le \eta \le 45\pi/32$$
$$47\pi/32 \le \eta \le 49\pi/32\,,\; 51\pi/32 \le \eta \le 53\pi/32$$
$$55\pi/32 \le \eta \le 57\pi/32\,,\; 59\pi/32 \le \eta \le 61\pi/32$$

we obtain

$$\Pr\{6\text{th bit correct}/s_i(t)\} = \Pr\{\cos 16\eta > 0/s_i(t)\} \tag{4.53}$$
$$= \int_{-\pi/32}^{\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{3\pi/32}^{5\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{7\pi/32}^{9\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{11\pi/32}^{13\pi/32} f_{\eta/s_i(t)}(H/s_i)dH$$
$$+ \int_{15\pi/32}^{17\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{19\pi/32}^{21\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{23\pi/32}^{25\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{27\pi/32}^{29\pi/32} f_{\eta/s_i(t)}(H/s_i)dH$$
$$+ \int_{31\pi/32}^{33\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{35\pi/32}^{37\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{39\pi/32}^{41\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{43\pi/32}^{45\pi/32} f_{\eta/s_i(t)}(H/s_i)dH$$
$$+ \int_{47\pi/32}^{49\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{51\pi/32}^{53\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{55\pi/32}^{57\pi/32} f_{\eta/s_i(t)}(H/s_i)dH + \int_{59\pi/32}^{61\pi/32} f_{\eta/s_i(t)}(H/s_i)dH$$
$$i = 1,4,5,6,8$$

which can further be written as

$$= \int_{-\pi/32}^{\pi/32} [f_{\eta/s_i(t)}(H/s_i) + f_{\eta/s_i(t)}(H+\pi/2/s_i) + f_{\eta/s_i(t)}(H+\pi/s_i) + f_{\eta/s_i(t)}(H+3\pi/2/s_i)]dH$$
$$+ \int_{3\pi/32}^{5\pi/32} [f_{\eta/s_i(t)}(H/s_i) + f_{\eta/s_i(t)}(H+\pi/2/s_i) + f_{\eta/s_i(t)}(H+\pi/s_i) + f_{\eta/s_i(t)}(H+3\pi/2/s_i)]dH$$
$$+ \int_{7\pi/32}^{9\pi/32} [f_{\eta/s_i(t)}(H/s_i) + f_{\eta/s_i(t)}(H+\pi/2/s_i) + f_{\eta/s_i(t)}(H+\pi/s_i) + f_{\eta/s_i(t)}(H+3\pi/2/s_i)]dH$$

$$+\int_{11\pi/32}^{13\pi/32}[f_{\eta/s_i(t)}(H/s_i)+f_{\eta/s_i(t)}(H+\pi/2/s_i)+f_{\eta/s_i(t)}(H+\pi/s_i)+f_{\eta/s_i(t)}(H+3\pi/2/s_i)]dH$$

$$i = 1,4,5,8 \tag{4.54}$$

so that from the previous work, we can directly write

$$\begin{aligned}
&\Pr\{\text{6th bit correct}/s_i(t)\} = 1/2\exp(-E_s/N_o)\\
&+\sqrt{E_s/2\pi N_o/2}\,\Big[\int_{-\pi/32}^{\pi/32}[\exp(-E_s/N_o\sin^2(H-\alpha_i))\cos(H-\alpha_i)\\
&\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\}+\exp(-E_s/N_o\cos^2(H-\alpha_i))\\
&\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))\}]dH\\
&+\int_{3\pi/32}^{5\pi/32}[\exp(-E_s/N_o\sin^2(H-\alpha_i))\cos(H-\alpha_i)\\
&\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\}+\exp(-E_s/N_o\cos^2(H-\alpha_i))\\
&\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))\}]\,dH\\
&+\int_{7\pi/32}^{9\pi/32}[\exp(-E_s/N_o\sin^2(H-\alpha))\cos(H-\alpha_i)\\
&\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\}+\exp(-E_s/N_o\cos^2(H-\alpha_i))\\
&\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))\}]\,dH\\
&+\int_{11\pi/32}^{13\pi/32}[\exp(-E_s/N_o\sin^2(H-\alpha_i))\cos(H-\alpha_i)\\
&\{Q(-\sqrt{2E_s/N_o}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\cos(H-\alpha_i))\}+\exp(-E_s/N_o\cos^2(H-\alpha_i))\\
&\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_o}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_o}\sin(H-\alpha_i))\}]\,dH\,\Big]\\
&i = 1,4,5,8
\end{aligned}\tag{4.55}$$

and

$$\begin{aligned}
&\Pr\{\text{6th bit correct}/s_i(t)\} = 1/2\exp(-E_s/N_o)\\
&+\sqrt{E_s/2\pi N_o/2}\,\Big[\int_{-\pi/32}^{\pi/32}[\exp(-E_s/N_o\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})\\
&\{Q(-\sqrt{2E_s/N_o}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_o}\sin(H+\alpha_{17-i}))\}
\end{aligned}$$

$$+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$
$$\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]dH$$
$$+\int_{3\pi/32}^{5\pi/32}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$
$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}$$
$$+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$
$$\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]\,dH$$
$$+\int_{7\pi/32}^{9\pi/32}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$
$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}$$
$$+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$
$$\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]\,dH$$
$$+\int_{11\pi/32}^{13\pi/32}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$
$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}$$
$$+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$
$$\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]\,dH\,]$$
$$i = 9,12,13,16 \qquad (4.56)$$

Assume now that $s_i(t)$, $i=2,3,6,7,10,11,14,15$ is transmitted. From the explanations given on page 56, we have

$$\Pr\{\text{6th bit correct} / s_i(t)\} = \Pr\{\cos 16\eta < 0 / s_i(t)\} \qquad (4.57)$$
$$= 1 - \Pr\{\cos 16\eta > 0 / s_i(t)\}$$
$$i = 2,3,6,7,10,11,14,15$$

which can directly be written as

$$\Pr\{\text{6th bit correct}/s_i(t)\} = 1 - 1/2\exp(-E_s/N_0)$$
$$-\sqrt{E_s/2\pi N_0}/2\,[\int_{-\pi/32}^{\pi/32}[\exp(-E_s/N_0\sin^2(H-\alpha_i))\cos(H-\alpha_i)$$
$$\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\cos(H-\alpha_i))\}+\exp(-E_s/N_0\cos^2(H-\alpha_i))$$
$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_0}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\sin(H-\alpha_i))\}]dH$$

$$+\int_{3\pi/32}^{5\pi/32}[\exp(-E_s/N_0\sin^2(H-\alpha_i))\cos(H-\alpha_i)$$

$$\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\cos(H-\alpha_i))\}+\exp(-E_s/N_0\cos^2(H-\alpha_i))$$

$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_0}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\sin(H-\alpha_i))\}]\,dH$$

$$+\int_{7\pi/32}^{9\pi/32}[\exp(-E_s/N_0\sin^2(H-\alpha))\cos(H-\alpha_i)$$

$$\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\cos(H-\alpha_i))\}+\exp(-E_s/N_0\cos^2(H-\alpha_i))$$

$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_0}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\sin(H-\alpha_i))\}]\,dH$$

$$+\int_{11\pi/32}^{13\pi/32}[\exp(-E_s/N_0\sin^2(H-\alpha_i))\cos(H-\alpha_i)$$

$$\{Q(-\sqrt{2E_s/N_0}\cos(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\cos(H-\alpha_i))\}+\exp(-E_s/N_0\cos^2(H-\alpha_i))$$

$$\sin(H-\alpha_i)\{Q(-\sqrt{2E_s/N_0}\sin(H-\alpha_i))-Q(\sqrt{2E_s/N_0}\sin(H-\alpha_i))\}]\,dH\,]$$

$$i = 2,3,6,7 \tag{4.58}$$

and similarly

$$\Pr\{6\text{th bit correct}/s_i(t)\} = 1 - 1/2\exp(-E_s/N_0)$$

$$-\sqrt{E_s/2\pi N_0}/2\;[\int_{-\pi/32}^{\pi/32}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}$$

$$+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]dH$$

$$+\int_{3\pi/32}^{5\pi/32}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}$$

$$+\exp(-E_s/N_0\sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\cos(H+\alpha_{17-i}))\}]\,dH$$

$$+\int_{7\pi/32}^{9\pi/32}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})$$

$$\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}$$

$$
\begin{aligned}
&+\exp(-E_s/N_0\ \sin^2(H+\alpha_{17-i}))\cos(H+\alpha_{17-i})\\
&\{Q(-\sqrt{2E_s/N_0}\ \cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\ \cos(H+\alpha_{17-i}))\}]\ dH\\
&+\int_{11\pi/32}^{13\pi/32}[\exp(-E_s/N_0\cos^2(H+\alpha_{17-i}))\sin(H+\alpha_{17-i})\\
&\{Q(-\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\sin(H+\alpha_{17-i}))\}\\
&+\exp(-E_s/N_0\ \sin^2(H+\alpha_{174,\ \text{inus.i.}}))\cos(H+\alpha_{17-i})\\
&\{Q(-\sqrt{2E_s/N_0}\ \cos(H+\alpha_{17-i}))-Q(\sqrt{2E_s/N_0}\ \cos(H+\alpha_{17-i}))\}]\ dH\ ]\\
&\quad i = 10,11,14,15
\end{aligned}
\tag{4.59}
$$

From Equation 4.9 once again, we obtain

$$
\Pr\{\text{6th bit correct}\} = \frac{1}{16}\sum_{i=1}^{16}\Pr\{\text{6th bit correct}/s_i(t)\}
\tag{4.60}
$$

so that using Equations 4.55 and 4.58 and simplifying wherever possible, results in

$$
\begin{aligned}
&\Pr\{\text{6th bit in error}\} = 1/2-\sqrt{E_s/\pi N_0}\ [\int_0^{\pi/32}\ [\sum_{i=1}^{8}\xi_i\{\exp(-E_s/N_0\ \sin^2(H-\alpha_i))\\
&\cos(H-\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \cos(H-\alpha_i))\}\\
&+\exp(-E_s/N_0\ \cos^2(H-\alpha_i))\ \sin(H-\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \sin(H-\alpha_i))\}\\
&+\exp(-E_s/N_0\ \sin^2(H+\alpha_i))\ \cos(H+\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \cos(H+\alpha_i))\}\\
&+\exp(-E_s/N_0\ \cos^2(H+\alpha_i))\ \sin(H+\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \sin(H+\alpha_i))\}]dH\\
&+\int_{\pi/8}^{5\pi/32}\ [\sum_{i=1}^{8}\xi_i\{\exp(-E_s/N_0\ \sin^2(H-\alpha_i))\\
&\cos(H-\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \cos(H-\alpha_i))\}\\
&+\exp(-E_s/N_0\ \cos^2(H-\alpha_i))\ \sin(H-\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \sin(H-\alpha_i))\}\\
&+\exp(-E_s/N_0\ \sin^2(H+\alpha_i))\ \cos(H+\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \cos(H+\alpha_i))\}\\
&+\exp(-E_s/N_0\ \cos^2(H+\alpha_i))\ \sin(H+\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \sin(H+\alpha_i))\}]\ ]\ dH\\
&+\int_{\pi/4}^{9\pi/32}\ [\sum_{i=1}^{8}\xi_i\{\exp(-E_s/N_0\ \sin^2(H-\alpha_i))\\
&\cos(H-\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \cos(H-\alpha_i))\}\\
&+\exp(-E_s/N_0\ \cos^2(H-\alpha_i))\ \sin(H-\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \sin(H-\alpha_i))\}\\
&+\exp(-E_s/N_0\ \sin^2(H+\alpha_i))\ \cos(H+\alpha_i)\{1/8-1/4\ Q(\sqrt{2E_s/N_0}\ \cos(H+\alpha_i))\}
\end{aligned}
$$

$$+\exp(-E_s/N_o \cos^2(H+\alpha_i)) \sin(H+\alpha_i)\{1/8 - 1/4\ Q(\sqrt{2E_s/N_o}\ \sin(H+\alpha_i))\}]\ dH$$

$$+\int_{3\pi/8}^{13\pi/32} \Big[\sum_{i=1}^{8} \xi_i\{\exp(-E_s/N_o \sin^2(H-\alpha_i))$$

$$\cos(H-\alpha_i)\{1/8 - 1/4\ Q(\sqrt{2E_s/N_o}\ \cos(H-\alpha_i))\}$$

$$+\exp(-E_s/N_o \cos^2(H-\alpha_i)) \sin(H-\alpha_i)\{1/8 - 1/4\ Q(\sqrt{2E_s/N_o}\ \sin(H-\alpha_i))\}$$

$$+\exp(-E_s/N_o \sin^2(H+\alpha_i)) \cos(H+\alpha_i)\{1/8 - 1/4\ Q(\sqrt{2E_s/N_o}\ \cos(H+\alpha_i))\}$$

$$+\exp(-E_s/N_o \cos^2(H+\alpha_i)) \sin(H+\alpha_i)\{1/8 - 1/4\ Q(\sqrt{2E_s/N_o}\ \sin(H+\alpha_i))\}\Big]\ ] \equiv PE6$$

$$\equiv PE6 \qquad (4.61)$$

where ξ_i is defined as

$$\xi_i = \begin{cases} 1 & \text{when } i = 1,4,5,8 \\ -1 & \text{when } i = 2,3,6,7 \end{cases}$$

7. Composite Bit Error Rate

The expressions for the probability of each bit in error have been developed and now the composite bit error rate can be obtained by adding all individual bit error rates and averaging them to yield the following result, namely

$$\text{Bit Error Rate} = 1/6\,[PE1 + PE2 + PE3 + PE4 + PE5 + PE6] \equiv PE \qquad (4.62)$$

where the appropriate terms making up this equation are given by Equations 4.11, 4.15, 4.24, 4.39, 4.50, and 4.61.

This final expressions has been evaluated on the computer for what are known to be the optimum phase parameter values, that is, $\alpha_1 = \pi/64$, $\alpha_2 = 3\pi/64$, $\alpha_3 = 5\pi/64$, $\alpha_4 = 7\pi/64$, $\alpha_5 = 9\pi/64$, $\alpha_6 = 11\pi/64$, $\alpha_7 = 13\pi/64$ and $\alpha_8 = 15\pi/64$.

The numerical results in the form of a graph are presented in Figure 4.3, showing the individual bit error rates and the composite bit error rate. A discussion involving these numerical results and their significance is presented in the next chapter.

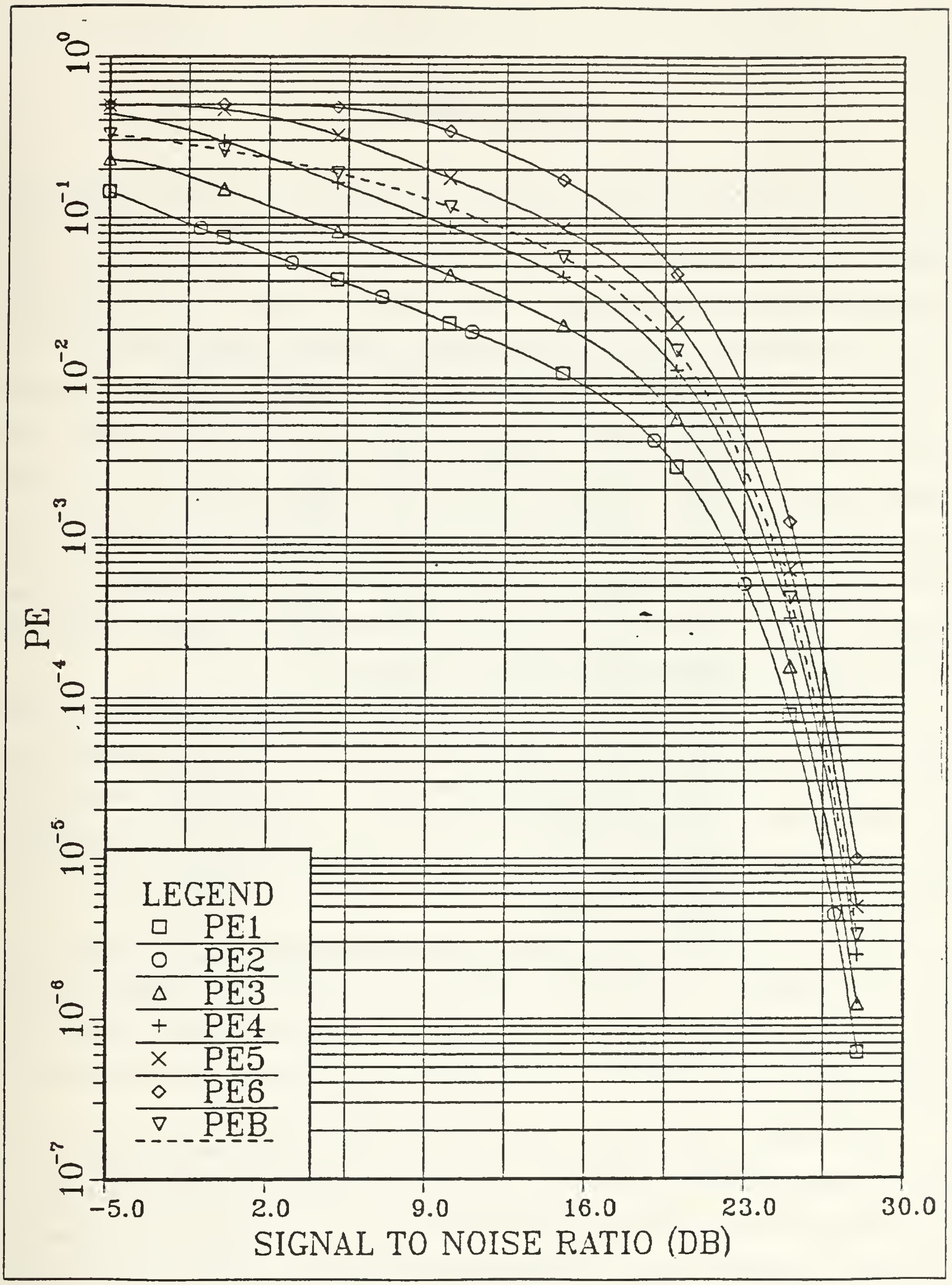

Figure 4.3 Receiver Noise Performance Curves for 64-PSK Modulated Signals.

V. CONCLUSIONS

The closed form expressions which were developed for the bit error probabilities in Chapters 3 and 4 were numerically evaluated on a computer as a function of signal-to-noise ratio. These results have been shown in the form of plots of bit error ratio versus signal-to-noise ratio in Figure 3.3 and in Figure 4.3 for the receivers designed to recover data transmitted via 32-PSK and 64-PSK modulated signals, respectively . The receivers have been implemented in such a way that the binary data (bits) are directly recovered. Numerical results have been obtained for optimum phase angle values. That is, for 32-PSK modulated signals, $\alpha = \pi/32$, $\beta = 3\pi/32$, $\gamma = 5\pi/32$ and $\delta = 7\pi/32$ and for 64-PSK modulated signals $\alpha_1 = \pi/64$, $\alpha_2 = 3\pi/64$, $\alpha_3 = 5\pi/64$, $\alpha_4 = 7\pi/64$, $\alpha_5 = 9\pi/64$, $\alpha_6 = 11\pi/64$, $\alpha_7 = 13\pi/64$ and $\alpha_8 = 15\pi/64$. These angle values were obtained from known results on the performance of receivers for 8-PSK and 16-PSK modulated signals [see Refs. 1,2], and the well-known fact that optimum receiver performance occurs at equal phase angle spacing between the signals.

The numerically evaluated performance results were compared with those of the standard phase measurement receiver and were found to be in complete agreement with the bit error rate expressions evaluated in Reference 6. These numerical values of bit error probabilities in the form of tables are presented in Figure 5.1 for the standard phase measurement receivers and in Figure 5.2 for Direct Bit Detection receivers for 32-PSK and 64-PSK modulated signals.

Comparison of the two tables reveals that the numerical values agree with each other. An exception is the 32-PSK case with E_b/N_0 equal to 25 dB, where the table entries are not quite in agreement. This difference can probably be attributed to different methods of computer numerical evaluations and the fact that at large values of SNR, problems with accuracy and round-off error begin to appear.

These results however indicate that the theoretical noise performance of the direct bit detection receivers is similar to that of standard phase measurement receivers. This establishes the fact that direct bit detection receivers are optimum in average bit error sense. Furthermore, by carefully structuring the receiver, parallel decoding of bits can be obtained with relatively simple hardware implementations.

EXACT BER VALUES OF *M*-ARY PSK WITH GRAY MAPPING

E_b/N_0, dB	P_b (M = 2,4)	P_b (M = 8)	P_b (M = 16)	P_b (M = 32)	P_b (M = 64)
−5.0	2.132E-01	2.468E-01	2.867E-01	3.185E-01	3.376E-01
−4.0	1.861E-01	2.217E-01	2.646E-01	3.003E-01	3.242E-01
−3.0	1.584E-01	1.961E-01	2.420E-01	2.817E-01	3.099E-01
−2.0	1.306E-01	1.708E-01	2.191E-01	2.629E-01	2.951E-01
−1.0	1.038E-01	1.461E-01	1.965E-01	2.442E-01	2.800E-01
0.0	7.865E-02	1.227E-01	1.745E-01	2.256E-01	2.649E-01
1.0	5.628E-02	1.008E-01	1.535E-01	2.073E-01	2.497E-01
2.0	3.751E-02	8.061E-02	1.338E-01	1.892E-01	2.345E-01
3.0	2.288E-02	6.225E-02	1.155E-01	1.714E-01	2.194E-01
4.0	1.250E-02	4.589E-02	9.865E-02	1.538E-01	2.043E-01
5.0	5.954E-03	3.186E-02	8.292E-02	1.368E-01	1.895E-01
6.0	2.388E-03	2.048E-02	6.816E-02	1.207E-01	1.747E-01
7.0	7.727E-04	1.195E-02	5.429E-02	1.055E-01	1.599E-01
8.0	1.909E-04	6.181E-03	4.145E-02	9.147E-02	1.452E-01
9.0	3.363E-05	2.748E-03	2.998E-02	7.840E-02	1.307E-01
10.0	3.872E-06	1.011E-03	2.025E-02	6.614E-02	1.165E-01
11.0	2.613E-07	2.937E-04	1.256E-02	5.451E-02	1.030E-01
12.0	9.006E-09	6.338E-05	7.010E-03	4.349E-02	9.027E-02
13.0		9.417E-06	3.427E-03	3.325E-02	7.848E-02
14.0		8.756E-07	1.421E-03	2.406E-02	6.752E-02
15.0		4.516E-08	4.789E-04	1.627E-02	5.424E-02
16.0			1.246E-04	1.010E-02	4.747E-02
17.0			2.342E-05	5.642E-03	3.819E-02
18.0			2.925E-06	2.763E-03	2.950E-02
19.0			2.187E-07	1.147E-03	2.163E-02
20.0			8.573E-09	3.876E-04	1.486E-02
21.0				1.011E-04	9.417E-03
22.0				1.907E-05	5.394E-03
23.0				2.393E-06	2.725E-03
24.0				1.799E-07	1.177E-03
25.0				7.099E-09	4.176E-04
26.0					1.159E-04
27.0					2.361E-05
28.0					3.264E-06

Figure 5.1 Tables of BER for Standard Phase Measurement Receiver [5].

E_b/N_0 db	32-PSK	64-PSK
-5.000	0.313904040466897238	0.326676360932059429
-4.000	0.296410028204559625	0.315427185798479059
-3.000	0.278585636661094399	0.302745107081563575
-2.000	0.260630227587930960	0.289121006314289969
-1.000	0.242678892447510092	0.274987963933914670
0.0000	0.224781610339377574	0.260645113916784646
1.000	0.206921945205848015	0.246249487232107891
2.000	0.189080319309712419	0.231867705599741014
3.000	0.171314097515894756	0.217527731091949228
4.000	0.153802590770912795	0.203222951468598367
5.000	0.136821451486116019	0.188894515491021833
6.000	0.120659373237207712	0.174455588433373748
7.000	0.105522645698072348	0.159866166953125097
8.000	0.914699187474389830E-01	0.145199605361483228
9.000	0.784049535807069264E-01	0.130649900458081503
10.00	0.661399350991582619E-01	0.116482403733065704
11.00	0.545117211422651805E-01	0.102959798006905634
12.00	0.434950037025787798E-01	0.902707191040171691E-01
13.00	0.332476326614213266E-01	0.784803529472881817E-01
14.00	0.240632941729371479E-01	0.675204926113723014E-01
15.00	0.162662212303958499E-01	0.572351720679376398E-01
16.00	0.101001998667220837E-01	0.474747877179158119E-01
17.00	0.564242309691505121E-02	0.381946254317088981E-01
18.00	0.276274436687314886E-02	0.295030325634878330E-01
19.00	0.114726723020487219E-02	0.216337539022855938E-01
20.00	0.387627183869726770E-03	0.148630493581698555E-01
21.00	0.101134556598161788E-03	0.941685878475830105E-02
22.00	0.190824504970248479E-04	0.539443103077843263E-02
23.00	0.240335895203930837E-05	0.272540871123795820E-02
24.00	0.190270994046229485E-06	0.117699363117420219E-02
25.00	0.174743203788420312E-07	0.417687244394825314E-03
26.00		0.115899852772443128E-03
27.00		0.236262136340722792E-04
28.00		0.327803243531177570E-05

Figure 5.2 Tables of BER for DBD Receivers for 32-PSK and 64-PSK Modulated Signals.

It is noteworthy that higher data rates or narrower signal bandwidths in higher order MPSK modulated signals is achieved at the expense of signal-to-noise ratio. Observation of the performance curves for each bit reveals that first (MSB) and second bits have the lowest bit error rates for the same signal-to-noise ratio and the BER increases for the third, fourth, fifth and sixth bit gradually. To equalize the individual bit error rates for each bit is a complex problem. For each value of SNR, there is a set of angle values (α, β , γ and δ for 32-PSK and α_i, $i = 1,2,\text{-------},8$, for 64-PSK) that equalizes the indvidual error ratio. Solving for these angle values requires a very complex computer search which is of little value, as with additional hardware it is possible to accomplish this BER equalization directly, without prior knowledge of the operating SNR values. However, PE $\sim$ PE1 in most practical situations (which involve operating BER values greater than 10^{-4}).

It can also be observed that for the same receiver performance level, 6 dB more signal-to-noise ratio is required for 64-PSK than for 32-PSK modulated signals. The receiver structures shown indicate that for 32-PSK modulated signals, the receiver is a special case of the receiver for 64-PSK modulated signals. This means that if power is the limiting factor, 32-PSK modulated signals should be used, while if bandwidth is the limiting factor, 64-PSK modulated signals should be used to transmit data. It can be shown that for 32-PSK modulated signals

$$BW = 2/T_s = 2/5T_b$$

and for 64-PSK mdulated signals

$$BW = 2/T_s = 2/6T_b$$

therefore, the bandwidth reduction factor is 5/6, while power increase factor is 4.

These results are important considering the present trend of using higher order modulation techniques due to ever increasing demands for higher data rates in digital transmission.

The advantages of the direct bit detection receivers can now be summarized in light of the work carried out in this thesis, namely

(1) Each bit of the transmitted symbol is recovered independently. Thus, the receiver also functions as a deinterleaver in those situations where the composite bit stream is created by interleaving two, three or more individual bit streams. This dispenses with the complex cicuitry required in the standard phase measurement receiver for regenerating bits from symbols.

(2) Same receiver noise performance as for standard phase measurement receivers is a remarkable result in light of the proposed simple hardware implementation.

(3) Each bit of the symbol is recovered using voltage comparisons with a zero threshold. This means that receiver performance remains robust during fading signal conditions, and the need for the transmission of training sequences and for the adjustment of AGC circuitry is eliminated.

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